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FIBONACCI LENGTHS INVOLVING THE WALL NUMBER k(n)

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Abstract. Two infinite classes of special finite groups considered (The group G is special, if G' and $Z(G)$ coincide). Using certain sequences of numbers we give explicit formulas for the Fibonacci lenghts of these classes which involve the well-known Wall numbers k(n).

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1. Introduction

Many authors have studied the periodic sequences of elements of finite groups. It was then refined in [4] and [5] and to that of a Fibonacci length of a given group $G = \langle a_1, a_2, \ldots, a_n \rangle$. Since 1990 the Fibonacci length has been studied and calculated for certain classes of finite groups (for example one may see [2,4,5,7]) where the least positive integer l is called the Fibonacci length of the group $G = \langle A \rangle$ with respect to the generating set $A = \{a_1, \ldots, a_n\}$, denoted by $LEN_A(G)$, if it is the period of the sequence $x_i = a_i, (1 \le i \le n), x_{n+i} = \prod_{j=1}^{n} x_{i+j-1}, i \ge 1$, of the elements of G. When it is clear which generating set is being investigated we will write $LEN(G)$ for $LEN_A(G)$. The Fibonacci sequence $\{f_n\}_{-\infty}^{\infty}$ of numbers defined by $f_n = f_{n-2} + f_{n-1}$ for $n \ge 0$, and $f_n = f_{n+2} - f_{n+1}$ for $n \le 0$, and we seed the sequence with $f_0 = 0$ and $f_1 = 1$. We use $k(n)$ to denote the fundamental period of the sequence, and call it the Wall number (see [9]). Using the Theorem 5 of [9] and the Wall's conjecture (see [2,9]), for a prime p and the integer $\alpha \ge 1$, we get $k(p^\alpha) = p^{\alpha-1}k(p)$. Also we define the sequence $\{g_n\}_0^\infty$ by

$$g_n = g_{n-1} + g_{n-2} + f_{n-2}^2, (n \ge 3), \; g_0 = g_1 = g_2 = 0.$$

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The standard notations $[a,b]=a^{-1}b^{-1}ab$ (for the commutator of the elements a and b of a group) and $|G:H|$ (the index of the subgroup H in the group G) have been used through this paper. We consider two classes of finitely presented groups as:

$$G_{mn}=\langle a,b|a^m=b^n=1,\ [a,b]^a=[a,b],\ [a,b]^b=[a,b]\rangle,\ m,n\geq 1,$$

$$H_m=\langle x,y|x^{m^2}=y^m=1,y^{-1}xy=x^{1+m}\rangle,\ m\geq 2.$$

In the Sections 2 and 3 we study the groups G_{mn} and show that the Fibonacci length of $G_n=G_{nn}$ may be defined explicitly in terms of the Wall number $k(n)$. Section 4 is devoted to study of the groups H_m.

2. A presentation for $Z(G_{mn})$

In this section, we give a presentation for $Z(G_{mn})$, the center of the group G_{mn}, and hereby we show that G_{nn} is an extra special finite group. This computation is almost similar to the counting of the centralizer of finite groups which for certain groups has been done in [1]. However our calculation involves an infinite class of finite groups. First we show that G_{mn} is finite for every positive integers m and n. Indeed we show that $|G_{mn}|=g.c.d.(m,n)\times mn$.

Consider the subgroup $H=<a,[a,b]>$ of G_{mn}. Obviously H is abelian and a simple coset enumeration by defining n cosets as $1=H$ and $ib=i+1,\ 1\leq i\leq n-1$ shows that $|G:H|=n$. Using the modified Todd-coxeter coset enumeration algorithm in the form given in [2] yields the following presentation for H:

$$H=\langle h_1,h_2|\ h_1^m=h_2^m=h_1^n=h_2^n=[h_1,h_2]=1\ \rangle.$$

If $d=g.c.d.(m,n)$, then $H\simeq Z_m\times Z_d$. So $|G_{mn}|=d\times mn$ as desired.

Proposition 2.1. *Let $G=G_{mn}$. Then $Z(G)$, the center of G, has a presentation isomorphic to*

$$Z(G)=\Big\langle x,y,z|x^{\frac{m}{d}}=y^{\frac{n}{d}}=z^d=[x,y]=[x,z]=[y,z]=1\Big\rangle.$$

Proof. Since $[a,b]^a=[a,b],\ [a,b]^b=[a,b]$, then $[a,b]\in Z(G)$ and also

$$[a,b^{-1}]=\left([a,b]^{b^{-1}}\right)^{-1}=[a,b]^{-1}\in Z(G),$$

$$[a^{-1},b]=\left([a,b]^{a^{-1}}\right)^{-1}=[a,b]^{-1}\in Z(G).$$

Then $G'=\langle[a,b]\rangle$ and $G'\subseteq Z(G)$. For, by the relations $[a,b]^a=[a,b]$ and $[a,b]^b=[a,b]$ of the group G and the well-known commutator relations $[xy,z]=[x,z]^y[y,z]$ and $[x,yz]=[x,z][x,y]^z$ which hold for every x,y,z in G, we conclude that every element $g\in G'$ is a power of $[a,b]$. (An almost easy calculation may be processed by letting every $g\in G'$ as a product of commutators $[x_i,y_i]$ where $x_i=z_1z_2\dots z_k,\ y_i=t_1t_2\dots t_p,\ z_i\in\{a,b\}$ or $z_i^{-1}\in\{a,b\}$, and $t_i\in\{a,b\}$

or $t_i^{-1} \in \{a,b\}$.) Since $|\frac{G}{G'}| = mn$, then $|G| = mnd$, where $d = g.c.d.(m,n)$. Consequently, $[a,b]^d = 1$.

Also, for every $x = x_1^{s_1} x_2^{s_2} \ldots x_k^{s_k}$ in G_{mn} where $x_i \in \{a,b\}$ and $s_1, s_2, \ldots, s_k$ are integers, and using the relations $b^j a^i = a^i b^j [b^j, a^i]$, we may easily prove that every element of G is in the form $a^i b^j g$ where $0 \leq i \leq m-1$, $0 \leq j \leq n-1$ and $g \in G'$ (by an induction method on the length of the word x).

To drive a generating set for $Z(G_{mn})$, suppose $x \in Z(G_{mn})$. Then $a^i b^j \in Z(G_{mn})$ and $[a, b^j] = [a,b]^j = 1$ so $d|j$. Similarly $d|i$. Then $x = (a^d)^s (b^d)^t [a,b]^l$. Suppose

$$T = \left\langle a^d, b^d, [a,b] \ \mid (a^d)^{\frac{m}{d}} = (b^d)^{\frac{n}{d}} = [a,b]^d = [a^d, b^d] = 1 \right\rangle.$$

Then $Z(G_{mn}) \subseteq T$. Clearly, $a^d,\ b^d, [a,b] \in Z(G_{mn})$ and we get

$$T = Z(G_{mn}) \cong \left\langle x, y, z \ \mid x^{\frac{m}{d}} = y^{\frac{n}{d}} = z^d = 1, [x,y] = [x,z] = [y,z] = 1 \right\rangle.$$

□

As a result of 2.1 we see that G_{mn} is an extra special group ($G' \cong Z(G)$) if and only if $m = n$. So, considering the case $m = n$ is of interest. For small values of m and n ($m = n \leq 10$), these groups have been studied among the non-metacyclic groups of order less than 1000 (see [8]).

3. The Fibonacci length of G_{mn}

If $m = n$, then for the group $G_n = G_{nn}$ we know that $|G_n| = n^3$, $Z(G) = G'$, $|Z(G)| = n$, and every element of G_n can be written uniquely in the form $a^t b^s [a,b]^k$ where $0 \leq t, s, k \leq n-1$. For different values of m and n it is easy to see that $G'_{mn} \subseteq Z(G_{mn})$. Firstly we study the group G_n) and almost a similar result will be deduced for G_{mn}.

Lemma 3.1. *For every m ($m \geq 3$), every element x_m of the Fibonacci sequence of the group G_n can be written in the form $a^{f_{m-2}} b^{f_{m-1}} [b,a]^{g_{m-1}}$. And if LEN$G_n$)=t, then t is the least integer such that all of the equations*

$$\begin{cases} f_{t-1} \equiv 1 \pmod{n} \\ f_t \equiv 0 \pmod{n} \\ g_t \equiv 0 \pmod{n} \\ g_{t+1} \equiv 0 \pmod{n} \end{cases}$$

hold. Moreover, $k(n)$ divides $LEN(G_n)$, and for every integer m if $m|n$, then $LEN(G_m)$ divides $LEN(G_n)$.

Proof. We use an induction method on m. Indeed, $x_3 = a^{f_1} b^{f_2} [b,a]^{g_2}$, $x_4 = a^{f_2} b^{f_3} [b,a]^{g_3}$ and if $x_k = a^{f_{k-2}} b^{f_{k-1}} [b,a]^{g_{k-1}}$ ($4 \leq k \leq m-1$), then by the relation $x_m = x_{m-2}x_{m-1}$ we get

$$\begin{aligned} x_m &= x_{m-2}x_{m-1} = a^{f_{m-4}}b^{f_{m-3}}[b,a]^{g_{m-3}}a^{f_{m-3}}b^{f_{m-2}}[b,a]^{g_{m-2}} \\ &= a^{f_{m-4}}b^{f_{m-3}}a^{f_{m-3}}b^{f_{m-2}}[b,a]^{g_{m-3}+g_{m-2}} \end{aligned}$$

$$\begin{aligned} &= a^{f_{m-4}+f_{m-3}}b^{f_{m-3}+f_{m-2}}[b,a]^{g_{m-3}+g_{m-2}+f_{m-3}^2} \\ &= a^{f_{m-2}}b^{f_{m-1}}[b,a]^{g_{m-1}}. \end{aligned}$$

Using the above proof gives the proof of the second part. To complete the proof, let $t = LEN(G_n)$. Then

$$\begin{cases} f_{t-1} \equiv 1 \pmod{n}, \\ f_t \equiv 0 \pmod{n}, \\ g_t \equiv 0 \pmod{n}, \\ g_{t+1} \equiv 0 \pmod{n}. \end{cases}$$

So, the condition $m \mid n$ yields the equations

$$\begin{cases} f_{t-1} \equiv 1 \pmod{m}, \\ f_t \equiv 0 \pmod{m}, \\ g_t \equiv 0 \pmod{m}, \\ g_{t+1} \equiv 0 \pmod{m}. \end{cases}$$

Then the result follows immediately by letting $LEN(G_m)= s$, i.e.; $t = ks$ for some integer k. □

Lemma 3.2.

(i) *For every* $n \geq 3$, $g_n = \sum_{i=1}^{n-2} f_{n-i-1}f_i^2$.

(ii) *Let* $n = k(m)$. *If* $g_n \equiv 0 \pmod{m}$ *and* $g_{n+1} \equiv 0 \pmod{m}$, *then for every positive integer* t, $g_{tn} \equiv 0 \pmod{m}$ *and* $g_{tn+1} \equiv 0 \pmod{m}$ *respectively.*

Proof. For $n = 3$ and $n = 4$, we get

$$g_3 = g_1 + g_2 + f_1^2 = \sum_{i=1}^{1} f_{3-i-1}f_i^2$$

and

$$g_4 = f_1^2 + f_2^2 = \sum_{i=1}^{2} f_{4-i-1}f_i^2$$

respectively. Then by induction on n we get

$$\begin{aligned} g_n &= g_{n-1} + g_{n-2} + f_{n-2}^2 = \sum_{i=1}^{n-3} f_{n-i-2} f_i^2 + \sum_{i=1}^{n-4} f_{n-i-3} f_i^2 + f_{n-2}^2 \\ &= \sum_{i=1}^{n-3} (f_{n-i-2} + f_{n-i-3}) f_i^2 + f_{n-2}^2 = \sum_{i=1}^{n-3} f_{n-i-1} f_i^2 + f_{n-2}^2 \\ &= \sum_{i=1}^{n-2} f_{n-i-1} f_i^2. \end{aligned}$$

To prove (ii), clearly, $\sum_{i=1}^{n} f_{n-i-1} f_i^2 \equiv \sum_{i=1}^{n-2} f_{n-i-1} f_i^2 \equiv 0 (\mathrm{mod}\ m)$. Then

$$\begin{aligned} g_{tn} &= \sum_{i=1}^{tn-2} f_{tn-i-1} f_i^2 \\ &= \sum_{s=1}^{t-1} \left(\sum_{i=(s-1)n+1}^{sn} f_{tn-i-1} f_i^2 \right) + \sum_{i=(t-1)n+1}^{tn-2} f_{tn-i-1} f_i^2 \\ &\equiv (t-1) \sum_{i=1}^{n} f_{n-i-1} f_i^2 + \sum_{i=1}^{n-2} f_{n-i-1} f_i^2 \equiv 0 (\mathrm{mod}\ m). \end{aligned}$$

Similarly, $g_{tn+1} \equiv 0 (\mathrm{mod}\ m)$. □

Proposition 3.3. *For an integer $t \geq 1$ and for a prime p, let $n = p^t$. Then*

$$LEN(G_n) = \begin{cases} 6 \times 2^{t-1}, & \text{if } p = 2, \\ k(p^t), & \text{otherwise.} \end{cases}$$

Proof. Let $p = 2$. By induction on t we prove that $g_{2k(n)} \equiv 0 (\mathrm{mod}\ 2^t)$ and $g_{2k(n)+1} \equiv 0 (\mathrm{mod}\ 2^t)$. Indeed, $g_{2k(2)} = g_6 = \sum_{i=1}^{4} f_{6-i-1} f_i^2 \equiv 0 (\mathrm{mod}\ 2)$, and if $g_{2k(2^s)} \equiv 0 (\mathrm{mod}\ 2^s)$ $(1 \leq s \leq t-1)$, then we get

$$\begin{aligned} g_{2k(n)} &= \sum_{i=1}^{2k(n)-2} f_{2k(n)-i-1} f_i^2 \\ &= \sum_{i=1}^{k(n)} f_{2k(n)-i-1} f_i^2 + \sum_{i=k(n)+1}^{2k(n)-2} f_{2k(n)-i-1} f_i^2 \\ &\equiv 2 \sum_{i=1}^{k(n)} f_{k(n)-i-1} f_i^2 = 2 g_{2k(2^{t-1})} \equiv 0 (\mathrm{mod}\ 2^t). \end{aligned}$$

Similarly, $g_{2k(n)+1} \equiv 0(\text{mod } 2^t)$. Then by the Lemma 3.1 we get the results

$$k(2^t)|LEN(G_n),\ LEN(G_n)|2k(2^t).$$

To prove the equality $LEN(G_n) = 2k(2^t)$, it is sufficient to show that $g_{k(2^t)}$ is not a multiple of 2^t. Suppose $2^t|g_{k(2^t)}$. Then by the relation $g_{k(2^t)} \equiv 2g_{k(2^{t-1})}$ $(\text{mod } 2^t)$ we get consecutively:

$$g_{k(2^t)} \equiv 2g_{k(2^{t-1})} \equiv \cdots \equiv 2^{t-1}g_{k(2)} \equiv 2^{t-1}(\text{mod } 2^t).$$

So, $2^t|2^{t-1}$, which is a contradiction.

Now, let $p \geq 3$. Then,

$$x_{k(n)+1} = a^{f_{k(n)}-1}.b^{f_{k(n)}}.[b,a]^{g_{k(n)}} = a$$

$$\begin{aligned} x_{k(n)+2} &= a^{f_{k(n)}}.b^{f_{k(n)+1}}.[b,a]^{g_{k(n)+1}} \\ &= a^{f_0}.b^{f_1}.[b,a]^{g_{k(n)+1}} = b. \end{aligned}$$

For, $f_{k(p^t)+1} \equiv f_1 \equiv 1(mod\ p^t)$ and $f_{k(p^t)} \equiv f_0 \equiv 0(mod\ p^t)$. Then the relations $g_{k(p^t)+1} \equiv 0(mod\ p^t)$ and $g_{k(p^t)} \equiv 0(mod\ p^t)$ may be deduced by the results of [2,4]. Using 3.1 shows that $k(p^t)$ is a divisor of $LEN(G_n)$. So $LEN(G_n) = k(p^t)$. □

Note that the result of 3.3 may be generalized for $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_s^{\alpha_s}$ and in this case we have:

$$LEN(G_n)=\text{l.c.m.}(LEN(G_{p_1^{\alpha_1}}),\dots,LEN(G_{p_s^{\alpha_s}})).$$

Proposition 3.4. *For different positive integers m and n,*

$$LEN(G_{mn}) = l.c.m.(k(m),k(n)).$$

Proof. Let $v = l.c.m.(k(m),k(n))$, $u = LEN(G_{mn})$ and d be the order of $[a,b]$. As well as the Lemma 3.1 every element of the Fibonacci sequence of G_{mn} may be represented by $x_k = a^{f_{k-2}}b^{f_{k-1}}[b,a]^{g_{k-1}}$ for every $k \geq 3$. Then the equations $x_{u+1} = a$, $x_{u+2} = b$, $x_{v+1} = a$ and $x_{v+2} = b$ hold. Since $G'_{mn} \subseteq Z(G_{mn})$, then an almost tedious calculation shows that $[a,b]^{f_u} = 1$ and $[a,b]^{f_{u+1}-1} = 1$. So we get the key results $LEN(G_{mn})|v$ and $k(d)|u$ (by considering $f_u \equiv 0(mod\ d)$ and $f_{u+1} \equiv 1(mod\ d)$). The rest of proof is similar to that of 3.3 by considering the similar results of 3.2. □

4. The groups H_m, $m \geq 2$

Let $G = H_m$. An easy calculation shows that G' is cyclic and may be generated by the commutator [x,y]. Moreover, $|G'| = m$ and $G' = Z(G)$ (for, $y^rx^s \in Z(G)$ and then $m|r$,$m|s$; i.e., $y^rx^s = (x^m)^t = [x,y]^t \in G'$, and also $[[x,y],y] = 1$ and $[[x,y],x] = 1$). To calculate the Fibonacci length of H_m, we need some preliminary results.

For every $m \geq 2$ we define the sequence $\{t_n\}_3^\infty$ of numbers as follows:

$$t_3 = t_4 = 1 + m, \ldots, t_n = (1 + mf_{n-2})t_{n-2} + t_{n-1},\ n \geq 5.$$

And then,

Lemma 4.1.

(i) *Every element of the Fibonacci sequence of* H_m *may be represented by* $x_n = y^{f_{n-1}} x^{t_n}$.

(ii) *For every integers* $n \geq 5$ *and* $m \geq 2$,

$$t_n \equiv f_{n-2} + m\left(f_{n-2} + \sum_{i=1}^{n-4} f_i f_{2+i} f_{n-3-i}\right) (\text{mod } m^2).$$

Proof. Since $x_3 = xy = y^{f_2} x^{1+m}$, $x_4 = y^2 x^{1+m}$ and $x_n = x_{n-2} x_{n-1}$ then by an induction method we get:

$$\begin{aligned} x_n &= y^{f_{n-3}} x^{t_{n-2}}\ y^{f_{n-2}} x^{t_{n-1}} \\ &= y^{f_{n-3}+f_{n-2}}\ x^{((1+m)^{f_{n-2}}) \times t_{n-2} + t_{n-1}} \\ &= y^{f_{n-1}}\ x^{(1+mf_{n-2})t_{n-2}+t_{n-1}} \\ &= y^{f_{n-1}}\ x^{t_n} \end{aligned}$$

To prove (ii) we may also use an induction method. Verifying the assertion for $n = 5$ is easy and then we get

$$\begin{aligned} t_n &\equiv (1 + mf_{n-2})t_{n-2} + t_{n-1} \\ &\equiv f_{n-4} + mf_{n-4} + m\left(\sum_{i=1}^{n-6} f_i f_{i+2} f_{n-5-i}\right) + mf_{n-2} f_{n-4} + f_{n-3} \\ &\quad + mf_{n-3} + m\left(\sum_{i=1}^{n-5} f_i f_{i+2}\ f_{n-4-i}\right) \\ &\equiv f_{n-2} + mf_{n-2} + m\left(\sum_{i=1}^{n-4} f_i f_{i+2}\ f_{n-3-i}\right) (\text{mod } m^2). \end{aligned}$$

□

Lemma 4.2. *Let* $A_s = f_{s-2} + \sum_{i=1}^{s-4} f_i f_{i+2} f_{s-(i+3)}$ *and* $n = k(m),\ m \geq 2$. *Then*

$$\begin{aligned} A_{n+1} &= f_{n-1} + \sum_{i=1}^{n-3} f_i f_{i+2} f_{n-(i+2)} \\ &\equiv \sum_{i=1}^{n-2} f_i^2 f_{i+1} - 2\sum_{i=1}^{n-2} f_i f_{i+1}^2 + \sum_{i=1}^{n-1} f_i^3 (\text{mod } m), \end{aligned}$$

$$\begin{aligned} A_{n+2} &= f_n + \sum_{i=1}^{n-2} f_i f_{i+2} f_{n-(i+1)} \\ &\equiv \sum_{i=1}^{n-2} f_i f_{i+1}^2 - \sum_{i=1}^{n-2} f_{i+1} f_i^2 (\text{mod } m). \end{aligned}$$

Proof. Since for every $i \geq 1$, $f_{n-i} \equiv (-1)^{i+1} f_i (\text{mod } m)$, then

$$\begin{aligned} A_{n+1} &= f_{n-1} + \sum_{i=1}^{n-3} f_i f_{i+2} f_{n-(i+2)} \\ &\equiv f_{n-1} + \sum_{i=1}^{n-3} (-1)^{i+1} f_i f_{i+2}^2 \\ &\equiv f_{n-1} + \sum_{i=1}^{n-3} (-1)^{i+1} f_i (f_i + f_{i+1})^2 \\ &\equiv f_{n-1} + \sum_{i=1}^{n-3} (-1)^{i+1} (f_i f_{i+1}^2 + 2 f_i^2 f_{i+1} + f_i^3) \\ &\equiv f_{n-1} + \sum_{i=1}^{n-3} f_{i+1}^2 f_{n-i} - 2 \sum_{i=1}^{n-3} f_{n-(i+1)} f_i^2 + \sum_{i=1}^{n-3} f_{n-i}^3. \end{aligned}$$

However, expanding the summations yields :

$$\begin{aligned} \sum_{i=1}^{n-3} f_{i+1}^2 f_{n-i} &\equiv \sum_{i=2}^{n-2} f_{i+1} f_{n-i}^2 \equiv \sum_{i=2}^{n-2} f_i^2 f_{i+1} \equiv -1 + \sum_{i=1}^{n-2} f_i^2 f_{i+1}, \\ \sum_{i=1}^{n-3} f_{n-(i+1)} f_i^2 &\equiv \sum_{i=2}^{n-2} f_{n-(i+1)}^2 f_i \equiv \sum_{i=2}^{n-2} f_{i+1}^2 f_i \equiv \sum_{i=2}^{n-2} f_{i+1}^2 f_i - f_1 f_2^2 \\ &\equiv -1 + \sum_{i=1}^{n-2} f_{i+1}^2 f_i, \\ \sum_{i=1}^{n-3} f_{n-i}^3 &\equiv -2 + \sum_{i=3}^{n-1} f_i^3, \end{aligned}$$

respectively. So

$$A_{n+1} \equiv \left(\sum_{i=1}^{n-2} f_i^2 f_{i+1} - 2 \sum_{i=1}^{n-2} f_i f_{i+1}^2 + \sum_{i=1}^{n-1} f_i^3 \right) (\text{mod } m).$$

The second part may be proved in a similar way. □

Proposition 4.3. *For every $m \geq 2$, $LEN(H_m) = k(m^2)$.*

Proof. It is sufficient to show that for every prime p and every $\alpha \geq 1$ if $m = p^{\alpha}$, then $LEN(H_m) = k(m^2)$. By the Lemma 4.1-(i) we get $x_s = y^{f_{s-1}}.x^{t_s}$, where $t_s = f_{s-2} + m\left(f_{s-2} + \sum_{i=1}^{s-4} f_i f_{i+2} f_{s-(i+3)}\right)$. Let $n = k(m)$. Then by the Lemma 4.2,

$$
\begin{aligned}
A_{n+1} &\equiv \left(\sum_{i=1}^{n-2} f_i^2 f_{i+1} - 2\sum_{i=1}^{n-2} f_i f_{i+1}^2 + \sum_{i=1}^{n-1} f_i^3\right) (\text{mod } m),\\
A_{n+2} &= f_n + \sum_{i=1}^{n-2} f_i f_{i+2} f_{n-(i+1)} \equiv \sum_{i=1}^{n-2} f_i f_{i+1}^2 - \sum_{i=1}^{n-2} f_{i+1} f_i^2 (\text{mod } m),
\end{aligned}
$$

where, $A_s = f_{s-2} + \sum_{i=1}^{s-4} f_i f_{i+2} f_{s-(i+3)}$.

Now assume that $l = k(m^2) = k(p^{2\alpha}) = p^{\alpha} k(p^{\alpha})$. Then

$$
\begin{aligned}
A_{l+1} &= f_{l-1} + \sum_{i=1}^{l-3} f_i f_{i+2} f_{l-(i+2)}\\
&\equiv \sum_{i=1}^{l-2} f_i^2 f_{i+1} - 2\sum_{i=1}^{n-2} f_i f_{i+2}^2 + \sum_{i=1}^{l-1} f_i^3\\
&\equiv p^{\alpha}\left(\sum_{i=1}^{n-2} f_i^2 f_{i+1} - 2\ \sum_{i=1}^{n-2} f_i f_{i+1}^2 + \sum_{i=1}^{n-1} f_i^3\right) \equiv 0 (\text{mod } m),
\end{aligned}
$$

and similarly, $A_{l+2} \equiv 0 (\text{mod } m)$ (for, $f_{tk(m)+i} \equiv f_i (\text{mod } m), t \geq 1$). This yields

$$
t_{l+1} \equiv f_{l-1} (\text{mod } m^2),\ t_{l+2} \equiv 0 (\text{mod } m^2).
$$

So $x_{l+1} = y^{f_l}.x^{t_{l+1}} = x$ and $x_{l+2} = y$; i.e., $LEN(H_m) | k(m^2)$.

Let $v = LEN(H_m)$. Then we have

$$
\begin{cases}
f_v \equiv 0 (\text{mod } m),\\
f_{v+1} \equiv 1 (\text{mod } m),\\
f_{v-1} + mA_{v+1} \equiv 1 (\text{mod } m^2),\\
f_v + mA_{v+2} \equiv 0 (\text{mod } m^2).
\end{cases}
$$

So $v = tk(m)$ and this yields the equations

$$
\begin{cases}
f_{tk(m)-1} + mA_{tk(m)+1} \equiv 1 (\text{mod } m^2),\\
f_{tk(m)} + mA_{tk(m)+2} \equiv 0 (\text{mod } m^2).
\end{cases}
$$

Let $p \geq 3$. Then by the above properties of A_{n+1} and A_{n+2}, and the results of [2,4] we get

$$
\begin{cases}
A_{tk(m)+1} \equiv 0 (\text{mod } m),\\
A_{tk(m)+2} \equiv 0 (\text{mod } m).
\end{cases}
$$

Consequently, $f_{v-1} \equiv 1 (\text{mod } m^2)$ and $f_v \equiv 0 (\text{mod } m^2)$; i.e., $k(m^2) \mid v$. Then we get $LEN(H_m) = k(m^2)$.

Now let $m = 2^\alpha$, where $\alpha \geq 2$. Using the induction method gives us : $A_{k(m)+1} \equiv 2A_{k(2^{\alpha-1})+1} \equiv 2^{\alpha-1}(\text{mod } m)$. If t is even, then $A_{tk(m)+1} \equiv 0(\text{mod } m)$ (for, $A_{tk(m)+1} \equiv tA_{k(m)+1}(\text{mod } m)$) and the result follows similarly for $p \geq 3$.
Now assume t is odd. Then

$$A_{tk(m)+1} \equiv A_{k(m)+1} \equiv 2^{\alpha-1}(\text{ mod } m), f_{tk(m)-1} + \frac{m^2}{2} \equiv 1(\text{ mod } m^2)$$

and $f_{tk(m)} \equiv 0(\text{mod } m^2)$. So,

$$\begin{cases} f_{tk(m)-1} \equiv 1(\text{mod } 2^{2\alpha-1}), \\ f_{tk(m)} \equiv 0(\text{mod } 2^{2\alpha-1}). \end{cases}$$

And namely, $2^{2(\alpha-1)}k(2) \mid t2^{\alpha-1}k(2)$ which is a contradiction (for, t is odd). If $\alpha = 1$ then $LEN(H_2) = k(4) = 6$ may be checked by a simple hand calculation. □

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