Abstract

In this paper we present an explicit expression for the arbitrary positive integer powers of the tridiagonal Toeplitz matrices.

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1 Introduction

A tridiagonal Toeplitz matrix of order $n$ is given as

$$A = \begin{pmatrix} b & a \\ c & b & a \\ & \ddots & \ddots & \ddots \\ & c & b & a \\ & & c & b \end{pmatrix},$$

where $a \neq 0$ and $c \neq 0$. This type of matrices are appeared in a variety of application such as discretizing differential equations and for solving corresponding difference equations we need to compute the positive integer powers of this type of matrices. In [2], Rimas has considered this type of matrices with $a = c = 1$ and $b = 0$ and has given an expression for the positive integer powers of this type of matrices of odd order and in [3, 4] of even order. He also considered two another special cases in [5, 6]. In this paper we give an expression for the positive integer powers of matrices of the form (1) of arbitrary orders.

This paper is organized as follows. In section 2, an expression for the positive integer powers of the tridiagonal Toeplitz matrices is derived. Two examples are given in section 3.
2 Main results

First we recall the following lemma which gives the eigenvalues and eigenvectors of the tridiagonal Toeplitz matrices.

**Lemma 2.1** The eigenvalues and eigenvectors of tridiagonal Toeplitz matrix (1) are given by

\[
\lambda_j = b + 2a \sqrt{c/a} \cos\left(\frac{j\pi}{n+1}\right), \quad j = 1, 2, \ldots, n,
\]

and

\[
x_j = \begin{pmatrix}
(c/a)^{1/2} \sin(1j\pi/(n+1)) \\
(c/a)^{2/2} \sin(2j\pi/(n+1)) \\
(c/a)^{3/2} \sin(3j\pi/(n+1)) \\
\vdots \\
(c/a)^{n/2} \sin(nj\pi/(n+1))
\end{pmatrix}, \quad j = 1, 2, \ldots, n,
\]

i.e., \(Ax_j = \lambda_j x_j, \ j = 1, 2, \ldots, n\). Moreover the matrix \(A\) is diagonalizable and \(P = (x_1 \ x_2 \ \ldots \ x_n)\) diagonalizes \(A\), i.e., \(P^{-1}AP = D\), where \(D = \text{diag}(\lambda_1 \ \lambda_2 \ \ldots \ \lambda_n)\).

**Proof:** See [1].

By using this lemma we have \(A^n = PD^nP^{-1}\). Hence, it is enough to find an explicit expression for \(P^{-1}\). Let

\[
\tilde{D} = \text{diag}((c/a)^{1/2} \ (c/a)^{2/2} \ \ldots \ (c/a)^{n/2}).
\]

Then we have \(P = \tilde{D}\tilde{P}\), where \(\tilde{P} = (\tilde{x}_1 \ \tilde{x}_2 \ \ldots \ \tilde{x}_n)\) in which

\[
\tilde{x}_j = \begin{pmatrix}
\sin(1j\pi/(n+1)) \\
\sin(2j\pi/(n+1)) \\
\sin(3j\pi/(n+1)) \\
\vdots \\
\sin(nj\pi/(n+1))
\end{pmatrix}.
\]

Therefore, we have

\[
P^{-1} = \tilde{P}^{-1}\tilde{D}^{-1}.
\]

So an explicit expression for \(\tilde{P}^{-1}\) will suffice.

**Lemma 2.2** Suppose \(\tilde{P}\) is defined as above. Then

\[
\tilde{P}^{-1} = \frac{2}{n+1} \tilde{P}.
\]
Proof: To prove Eq. (7), it is enough to show that
\[ \tilde{x}_i^T \tilde{x}_j = \frac{n + 1}{2} \delta_{ij}, \quad i, j = 1, \ldots, n, \]  
(8)
where \( \delta_{ij} \) is the Kronecker delta notation. Taking relation (5) into account we have
\begin{align*}
\tilde{x}_i^T \tilde{x}_j &= \sum_{k=1}^{n} \sin \frac{ki \pi}{n+1} \sin \frac{kj \pi}{n+1} \\
&= \frac{1}{2} \sum_{k=1}^{n} \cos \frac{k(i-j) \pi}{n+1} - \cos \frac{k(i+j) \pi}{n+1}.
\end{align*}
(9)
Now if \( i = j \), then we have
\begin{align*}
\tilde{x}_i^T \tilde{x}_j &= \frac{n}{2} - \frac{1}{2} \sum_{k=1}^{n} \cos \frac{k(i+j) \pi}{n+1} \\
&= \frac{n}{2} \sin \frac{(2n+1)i \pi}{n+1} - \frac{1}{2}.
\end{align*}
In the latter equality, we have used the well-known identity
\[ \sum_{k=1}^{n} \cos k \theta = \frac{\sin(n+\frac{1}{2})\theta}{\sin \frac{\theta}{2}} - \frac{1}{2}, \]
(10)
Hence from
\[ \sin \frac{(2n+1)i \pi}{n+1} = -\sin \frac{i \pi}{n+1}, \]
we conclude that \( \tilde{x}_i^T \tilde{x}_j = \frac{n+1}{2} \). Let \( i \neq j \). Again, by using identity (10), Eq. (9) can be written as
\begin{align*}
\tilde{x}_i^T \tilde{x}_j &= \frac{1}{4} \left( \sin(n+\frac{1}{2})(i-j) \frac{\pi}{n+1} - \sin(n+\frac{1}{2})(i+j) \frac{\pi}{n+1} \right) \\
&= \frac{1}{4} \left( \frac{\sin((i-j)\pi) - (\frac{i-j}{2(n+1)})}{2(n+1)} - \frac{\sin((i+j)\pi) - (\frac{i+j}{2(n+1)})}{2(n+1)} \right).
\end{align*}
For all positive integer numbers \( i \) and \( j \), both \( i+j \) and \( i-j \) are even or both of them are odd. If both of them are even then
\begin{align*}
\sin((i-j)\pi) - (\frac{i-j}{2(n+1)}) &= \sin \frac{(i-j)\pi}{2(n+1)} , \\
\sin((i+j)\pi) - (\frac{i+j}{2(n+1)}) &= \sin \frac{(i+j)\pi}{2(n+1)}.
\end{align*}
Hence, in this case we have \( \tilde{x}_i^T \tilde{x}_j = 0 \). In the similar way if both of them are odd then we have \( \tilde{x}_i^T \tilde{x}_j = 0 \).
Theorem 2.3 Let $A$ be a tridiagonal Toeplitz matrix defined in (1) and $Z = A^m = (z_{ij})$, where $m$ is a positive integer. Then

$$z_{ij} = \frac{2}{n+1} \left(\frac{c}{a}\right)^{\frac{i-j}{2}} \sum_{k=1}^{n} \lambda_k^m \sin \frac{ik\pi}{n+1} \sin \frac{jk\pi}{n+1},$$

(11)

where $\lambda_k = b + 2a\sqrt{c/a} \cos \left(\frac{k\pi}{n+1}\right)$.

Proof: We have $A = P^{-1}DP$, where $P$ and $D$ were defined in Lemma 2.1. Hence

$$Z = A^m = PD^mP^{-1} = PD^m\tilde{P}^{-1}D^{-1}$$

using Eq. (6)

$$= \frac{2}{n+1}PD^m\tilde{P}\tilde{D}^{-1}$$

using Theorem 1

$$= \frac{2}{n+1}P \text{diag}(\lambda_1^m \lambda_2^m \ldots \lambda_n^m) \tilde{P} ((c/a)^{-1/2} (c/a)^{-2/2} \ldots (c/a)^{-n/2}).$$

Now by substituting $P$ and $\tilde{P}$ in the latter equation and a little computation the desired relation is obtained.

Remark. If all the eigenvalues of $A$ are nonzero ($A$ is nonsingular), then Theorem 2.3 is also valid for negative integer $m$. In the special case that $m = -1$, by using Theorem 2, the inverse of a tridiagonal Toeplitz matrix can be obtained.

3 Examples

Example 1. Let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{pmatrix}.$$

Here, we have $a = 1$, $b = 3$ and $c = 2$. By using Lemma 2.1 (Eq. 2), we have $\lambda_1 = 5$, $\lambda_2 = 3$ and $\lambda_3 = 1$. Let $m = 5$. Then from Theorem 2.3 we have

$$z_{ij} = 2^{i-j-2} \sum_{k=1}^{3} \lambda_k^5 \sin \frac{ik\pi}{4} \sin \frac{jk\pi}{4}, \quad i,j = 1,2,3.$$

By the latter relation we find

$$A^5 = \begin{pmatrix} 903 & 781 & 330 \\ 1562 & 1563 & 781 \\ 1320 & 1562 & 903 \end{pmatrix}.$$
Example 2. Let

\[
A = \begin{pmatrix}
5 & 1 & 0 & 0 \\
4 & 5 & 1 & 0 \\
0 & 4 & 5 & 1 \\
0 & 0 & 4 & 5
\end{pmatrix}.
\]

Here, we have \(a = 1\), \(b = 5\) and \(c = 4\). By using Lemma 2.1 (Eq. 2), we have \(\lambda_1 = 8.2361\), \(\lambda_2 = 6.2361\), \(\lambda_3 = 3.7639\) and \(\lambda_4 = 1.7639\). Let \(m = -1\). Then from Theorem 2.3 we have

\[
z_{ij} = \frac{1}{5} 2^{i-j+1} \sum_{k=1}^{4} \frac{1}{\lambda_k} \sin \frac{ik\pi}{5} \sin \frac{jk\pi}{5}, \quad i, j = 1, 2, 3, 4.
\]

By a little computations we obtain

\[
A^{-1} = \begin{pmatrix}
0.2493 & -0.0616 & 0.0147 & -0.0029 \\
-0.2463 & 0.3079 & -0.0733 & 0.0147 \\
0.2346 & -0.2933 & 0.3079 & -0.0616 \\
-0.1877 & 0.2346 & -0.2463 & 0.2493
\end{pmatrix}.
\]

References


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