Amenability of the restricted Fourier Stieltjes algebras

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An inverse semigroup $S$ is a discrete semigroup such that for each $s \in S$, there is a unique element $s^* \in S$ such that

$$ss^*s = s, \quad s^*ss^* = s^*.$$

One can show that $s \mapsto s^*$ is an involution on $S$ [7].
The semigroup algebra \( \ell^1(S) \) is a Banach algebra under convolution

\[
\ell^1(S) = \{ f : S \to \mathbb{C} : \sum_{s \in S} |f(s)| < \infty \}
\]

\[
f \ast g(x) = \sum_{st = x} f(s)g(t)
\]

and norm \( \|f\|_1 = \sum_{s \in S} |f(s)| \). However, there are several technical difficulties when one tries to do things on inverse semigroups similar to the group case. The main difficulty is that many well-known properties of the group algebra \( \ell^1(G) \) break down for inverse semigroups.
Difficulties

For instance the left regular representation $\lambda$ of an inverse semigroup looses its connection with positive definite functions. This is because the crucial equality

$$\langle \lambda(x^*)\xi, \eta \rangle = \langle \xi, \lambda(x)\eta \rangle$$

for $x \in S$ and $\xi, \eta \in \ell^2(S)$ fails in general. This makes it difficult to study the algebras defined based on positive definite functions [8], such as the Fourier and Fourier-Stieltjes algebras [4].
M. Amini and A. R. Medghalchi introduced and studied the notion of restricted semigroup algebra in [1], [2] to overcome such difficulties. They showed that if the convolution product on $\ell^1(S)$ is appropriately modified, one gets a Banach $\ast$-algebra $\ell^1_r(S)$, called the restricted semigroup algebra, which always has an approximate identity (not necessarily bounded).
In the new convolution product, positive definite functions fit naturally with a restricted version of the left regular representation $\lambda_r$. The idea is that we require the homomorphism property of representations to hold only for those pairs of elements in the semigroup whose range and domain match. This is quite similar to what is done in the context of groupoids, but the representation theory of groupoids is much more involved [2].
Restricted semigroup of $S$ which is denoted by $S_r$ has been defined by the multiplication rule

$$x \cdot y = \begin{cases} 
xy & \text{if } x^* x = yy^*, \\
0 & \text{otherwise}
\end{cases}$$

for $x, y \in S \cup \{0\}$.

For the definition of $\ell^1_r(S)$, $C^*_\lambda(S)$, $B_{r,e}(S)$ and $A_{r,e}(S)$ see [1]. In this paper we discuss the amenability of the restricted Fourier-Stieltjes algebras on inverse semigroups.
For each inverse semigroup $S$, states on the $*$-algebra $\mathbb{C}S$ (the vector space over $S$ spanned by $S$ with convolution and involution comes from $S$) are defined by D. Milan in [9]. It is a dense subalgebra of both the full and reduced $C^*$-algebras of $S$. A **state** on the algebra $\mathbb{C}S$ of an inverse semigroup $S$ is a positive linear map $\rho : \mathbb{C}S \rightarrow \mathbb{C}$ such that

$$\sup\{|\rho(a)|^2 : a \in \mathbb{C}S; \rho(a^*a) \leq 1\} = 1.$$ 

We know that

$$\|f\|_{C^*(S)} = \sup\{\rho(f^*f)^{1/2} : \rho \in S(C^*(S))\} = \sup\{\rho(f^*f)^{1/2} : \rho \in S(\mathbb{C}S)\}$$

(1)

where $f$ lies in $\mathbb{C}S$ and $S(A)$ denotes the set of states for the algebra $A$ [9]. When $S$ is a (discrete) group, all these concepts are already discussed by Eymard in [4].
The algebras $C_0^*(S)$ and $\mathbb{C}_0 S$, are just the quotients of the algebras with which we have been working by the ideal generated by the zero of $S$. 
A Fell bundle over a discrete group $G$ is a collection of closed subspaces $\mathcal{B} = \{B_g\}_{g \in G}$ of a $C^*$-algebra $\mathcal{B}$, satisfying $B_g^* = B_{g^{-1}}$ and $B_g B_h \subseteq B_{gh}$ for all $g$ and $h$ in $G$. 
As is the case with $C^*$-algebras, which can be defined, concretely, as a norm-closed *-subalgebra of $B(H)$, as well as a certain abstract mathematical object, defined via a set of axioms, Fell bundles may also be seen under a dual point of view, specially if one restricts attention to the case of discrete groups. The above definition of Fell bundles is the one we adopt here.
Also one defines its $\ell_1$ cross-sectional algebra denoted by $\ell_1(B)$, to be the Banach $*$-algebra consisting of the $\ell_1$ cross-sections of $B$ under certain multiplication, involution and norm, and the cross-sectional $C^*$-algebra of $B$, denoted by $C^*(B)$, is defined to be the enveloping $C^*$–algebra of $\ell_1(B)[5]$. 
In this lecture we denote the dual space of $C^*(\mathcal{B})$ by $B(\mathcal{B})$. 
Gradings

**Definition.** Let $S$ be an inverse semigroup with zero, a grading of $S$ by the group $G$ is a map $\varphi : S \rightarrow G \cup \{0\}$ such that $\varphi^{-1}(0) = \{0\}$ and $\varphi(ab) = \varphi(a)\varphi(b)$ provided that $ab \neq 0$.

Here we define the Fell bundle arising from a grading $\varphi$. For each $g \in G$, let

$$A_g = \text{span}\{s : \varphi(s) = g\} \text{ inside } C_0S, \quad B_g = \overline{A_g} \text{ inside } C_0^*(S).$$
By [9, Proposition 3.3] the collection $\mathcal{B} = \{B_g\}_{g \in G}$ is a Fell bundle for $C^*_0(S)$. Also again by [9] representations of $\mathcal{B}$ are in one-one correspondence with representations of $C^*_0(S)$ and hence $C^*(\mathcal{B})$ is isomorphic to $C^*_0(S)$.

Since $S_r$ is an inverse semigroup with zero, the grading map technique applies to $S_r$. 
Let $S$ be an inverse semigroup and $G$ its maximal homomorphic image group with $\varphi : S \rightarrow G$, we define the following new product on $G \cup \{0\}$. Put $\varphi(x) \circ \varphi(y) = \varphi(x \cdot y)$ and $\varphi(s^*) := \varphi(s^*)$. It is easy to see that this new multiplication makes $G \cup \{0\}$ an inverse semigroup, which will be denoted by $G^0$. 


Now there is a homomorphism $\varphi_r : S_r \to G^0$, $\varphi_r(s) = \varphi(s)$, $s \in S$ and $\varphi_r(0) = 0$ that induces the natural map $\theta : \mathbb{C}S_r \to \mathbb{C}G^0$ by

$$\theta\left(\sum_{s \in S_r} \alpha_s s\right) = \sum_{s \in S_r} \alpha_s \varphi_r(s)$$
Proposition. For each $a \in \mathbb{C}S_r$, $\|\theta a\|_{C^*(G^0)} \leq \|a\|_{C^*(S_r)}$. 
A strongly $E$-unitary inverse semigroup $S$ is an inverse semigroup that admits a grading $\phi : S \longrightarrow G \cup 0$ such that $\phi^{-1}(e)$ is equal to the set of nonzero idempotent of $S$, where $e$ is the identity of $G$. 
Lemma. Let $S$ be an $E$-unitary inverse semigroup, then $S_r$ is strongly $E^*$–unitary inverse semigroup.

Proposition. Let $S$ be an $E$–unitary inverse semigroup with the maximal homomorphomic group image $G$. Then the natural map $\theta : \mathbb{C}S_r \to \mathbb{C}G^0$ is an isometry.
**Lemma.** Let $S$ be an $E$-unitary inverse semigroup, then $S_r$ is strongly $E^*$–unitary inverse semigroup.

**Proposition.** Let $S$ be an $E$–unitary inverse semigroup with the maximal homomorphic group image $G$. Then the natural map $\theta : \mathbb{C}S_r \longrightarrow \mathbb{C}G^0$ is an isometry.
The Main Theorem

Theorem.
Let \( \varphi : S \to G \) be the epimorphism of the \( E \)-unitary inverse semigroup \( S \) onto its maximal homomorphic image. Then there exists an isometric isomorphism \( \phi : B(\mathcal{B}) \to B_{r,e}(S) \) such that for each \( f \) in \( B(\mathcal{B}) \), \( \phi(f) = f \circ \varphi \), where \( \mathcal{B} \) is the Fell bundle for \( C^*_0(G^0) \).

*** So in this case, the amenability of \( B_{r,e}(S) \) is equivalent to the amenability of \( B(\mathcal{B}) \).
**Definition.** The inverse semigroup $S$ has restricted weak containment property if $C^*_r(S) \cong_{iso} C^*_\lambda(S)$.

**Proposition.** For an inverse semigroup $S$, $S_r$ has weak containment property if and only if $S$ has restricted weak containment property.
**Definition.** The inverse semigroup $S$ has restricted weak containment property if $C_{r}^{*}(S) \cong_{iso} C_{\lambda r}^{*}(S)$.

**Proposition.** For an inverse semigroup $S$, $S_{r}$ has weak containment property if and only if $S$ has restricted weak containment property.
Proposition. For an inverse semigroup $S$, $\ell_r^1(S)$ is amenable if and only if $E_S$ is finite, if and only if $\ell_r^1(S)$ has a bounded approximate identity.

Theorem. Let $H = \varphi^{-1}(e) \leq S$ where $\varphi$ is the quotient map of $S$ onto its maximal group homomorphic image. Then $S$ has restricted weak containment property if and only if $\varepsilon : C^*_r(S) \to C^*_r(H)$ is faithful and $H$ has restricted weak containment property.

Corollary. An $E$-unitary inverse semigroup $S$ has restricted weak containment property if and only if $\varepsilon : C^*_r(S) \to C^*_r(E)$ is faithful if and only if the Fell bundle of $C^*_r(S)$ is amenable.
Proposition. For an inverse semigroup $S$, $\ell^1_r(S)$ is amenable if and only if $E_S$ is finite, if and only if $\ell^1_r(S)$ has a bounded approximate identity.

Theorem. Let $H = \varphi^{-1}(e) \leq S$ where $\varphi$ is the quotient map of $S$ onto its maximal group homomorphic image. Then $S$ has restricted weak containment property if and only if $\varepsilon : C^*_r(S) \rightarrow C^*_r(H)$ is faithful and $H$ has restricted weak containment property.

Corollary. An $E$-unitary inverse semigroup $S$ has restricted weak containment property if and only if $\varepsilon : C^*_r(S) \rightarrow C^*_r(E)$ is faithful if and only if the Fell bundle of $C^*_r(S)$ is amenable.
**Proposition.** For an inverse semigroup $S$, $\ell_r^1(S)$ is amenable if and only if $E_S$ is finite, if and only if $\ell_r^1(S)$ has a bounded approximate identity.

**Theorem.** Let $H = \varphi^{-1}(e) \leq S$ where $\varphi$ is the quotient map of $S$ onto its maximal group homomorphic image. Then $S$ has restricted weak containment property if and only if $\varepsilon : C_r^*(S) \to C_r^*(H)$ is faithful and $H$ has restricted weak containment property.

**Corollary.** An $E$-unitary inverse semigroup $S$ has restricted weak containment property if and only if $\varepsilon : C_r^*(S) \to C_r^*(E)$ is faithful if and only if the Fell bundle of $C_r^*(S)$ is amenable.
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References


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Introduction
Main Results
Acknowledgements
The Bibliography

Marzieh Shams Yousefi
Amenability of the restricted Fourier Stieltjes algebras