Sets

Definition of a set

The definition of a set sounds very vague at first. A set can be defined as a collection of things that are brought together because they obey a certain rule. These ‘things’ may be anything you like: numbers, people, shapes, cities, bits of text …., literally anything.

The key fact about the ‘rule’ they all obey is that it must be well-defined. In other words, it enables us to say for sure whether or not a given ‘thing’ belongs to the collection. If the ‘things’ we’re talking about are English words, for example, a well-defined rule might be: ‘…has 5 or more letters’

A rule which is not well-defined (and therefore couldn’t be used to define a set) might be: ‘…is hard to spell’

Elements

A ‘thing’ that belongs to a given set is called an element of that set. For example:

HENRY VIII is an element of the set of Kings of England.

Notation

‘Curly brackets’ – braces – {…} - are used to stand for the phrase ‘the set of …’. These braces can be used in various ways. For example:

• We may list the elements of a set:
{−3, −2, −1, 0, 1, 2, 3}

• We may describe the elements of a set:
{integers between -3 and 3 inclusive}

• We may use an identifier (the letter x for example) to represent a typical element, a ‘|’ symbol to stand for the phrase ‘such that’, and then the rule or rules that the identifier must obey:
{x | x is an integer and |x| < 4}

The last way of writing a set – called set comprehension notation – can be generalized as:
{x | P(x)}, where P(x) is a statement (technically a propositional function) about x and the set is the collection of all elements x for which P is true.

The Greek letter is used as follows:

• \(\in\) means ‘is an element of …’. For example: dog \(\in\) {quadrapeds}

• \(\notin\) means ‘is not an element of …’. For example: washington DC \(\notin\) {European capital cities}

A set can be finite: {British citizens} … or infinite: {7, 14, 21, 28, 35, …}

(Note the use of the ellipsis -…- to indicate that the sequence of numbers continues indefinitely)
Some Special Sets

Universal Set
The set of all the ‘things’ currently under discussion is called the universal set (or sometimes, simply universe). It is denoted by U.
The universal set doesn’t contain everything in the whole universe. On the contrary, it restricts us to just those things that are relevant at a particular time. For example, if in a given situation we’re talking about numeric values – quantities, sizes, times, weights, or whatever – the universal set will be a suitable set of numbers (see below). In another context, the universal set may be {alphabetic characters} or {all living people}, etc.

Null set (empty set)
The set containing no elements at all is called the null set, or empty set. It is denoted by a pair of empty braces: {}, or by the symbol ∅. It may seem odd to define a set that contains no elements. Bear in mind, however, that one may be looking for solutions to a problem where it isn’t clear at the outset whether or not such solutions even exist. If it turns out that there isn’t a solution, then the set of solutions is empty.

Some special sets of numbers
Several sets are used so often, they are given special symbols.

The natural numbers
The ‘counting’ numbers (or whole numbers) starting at 1, are called the natural numbers. This set is sometimes denoted by N. So \( N = \{1, 2, 3, \ldots\} \).

Integers
All whole numbers, positive, negative and zero form the set of integers. It is sometimes denoted by \( \mathbb{Z} \). so \( \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \).

Real numbers
If we expand the set of integers to include all decimal numbers, we form the set of real numbers. The set of real is sometimes denoted by \( \mathbb{R} \).
A real number may have a finite number of digits after the decimal point (e.g. 3.625), or an infinite number of decimal digits. In the case of infinite number of digits, these digits may:
- recur, e.g. 8.127127127...
- \( \ldots \) or they may not recur; e.g.3.141592653...

Rational numbers
Those real numbers whose decimal digits are finite in number, or which recur, are called rational numbers. The set of rationals is sometimes denoted by the letter \( \mathbb{Q} \).
A rational number can always be written as exact fraction \( p/q \), where \( p \) and \( q \) are integers.
- For example: 0.5, -17, 2/17, 82.01, 3.282828… are all rational numbers.

Irrational numbers
If a number can’t be represented exactly by a fraction \( p/q \), it is said to be irrational.
- Examples include: \( \sqrt{2}, \sqrt{3}, \pi \).
Relationships between Sets
We’ll now look at various ways in which sets may be related to one another.

Equality
Two sets A and B are said to be equal if and only if they have exactly the same elements. In this case, we simply write: \( A = B \).

Note two further facts about equal sets:
- The order in which elements are listed does not matter.
- If an element is listed more than once, any repeat occurrences are ignored.
So, for example, the following sets are all equal:
\{1,2,3\} = \{3,2,1\} = \{1,1,2,3,2,2\}

Subsets
If all the elements of a set A are also elements of a set B, then we say that A is a subset of B, and we write: \( A \subseteq B \)
For example:
If \( T = \{2,4,6,8,10\}\) and \( E = \{\text{even integers}\} \), then \( T \subseteq E \).
Notice that \( A \subseteq B \) does not imply that B must necessarily contain extra elements that are not in A; the two sets could be equal.
However, if, in addition, B contains at least one element that isn’t in A, then we say that A is a proper subset of B. In such a case we would write:
\( A \subset B \)
Notice also that every set is a subset of the universal set, and the empty set is a subset of every set. (You might be curious about this last statement; how can the empty set be a subset of anything, when it doesn’t contain any elements? The point here is that for every set A, the empty set \( \emptyset \) doesn’t contain any elements that aren’t in A. So \( \emptyset \subseteq A \) for all sets A.)
Finally note that if \( A \subseteq B \) and \( B \subseteq A \) then A and B must contain exactly the same elements, and are therefore equal. In other words:
- \( A \subseteq B \) and \( B \subseteq A \) \( \Rightarrow A = B \)

Disjoint
Two sets are said to be disjoint if they have no elements in common. For example:
- If \( A = \{\text{even numbers}\} \) and \( B = \{1,3,5,11,19\} \), then A and B are disjoint.

Venn Diagrams
A Venn diagram can be a useful way of illustrating relationships between sets.
In a Venn diagram:
- The universal set is represented by a rectangle. Points inside the rectangle represent elements that are in the universal set; points outside represent things not in the universal set. You can think of this rectangle, then, as a ‘fence’ keeping unwanted things out – and concentrating our attention on the things we’re talking about.
- Other sets are represented by loops, usually oval or circular in shape, drawn inside the rectangle. Again, points inside a given loop represent elements in the set it represents; points outside represent things not in the set.

![Fig. 1: Venn diagrams](image1)
![Fig. 2: Venn diagrams](image2)
On the left, the sets A and B are disjoint, because the loops don’t overlap. On the right A is a subset of B, because the loop representing set A is entirely enclosed by loop B.

**Operations on Sets**

**Intersection**
We can define intersection as follows:
- The **intersection** of two sets A and B, written \( A \cap B \), is the set of elements that are in A and in B. (Note that in symbolic logic, a similar symbol, \( \land \), is used to connect two logical propositions with the AND operator.) For example, if \( A = \{1,2,3,4\} \) and \( B = \{2,4,6,8\} \), then \( A \cap B = \{2,4\} \).

We can say, then that we have combined two sets to form a third set using the operation of intersection.

**Union**
In a similar way we can define the union of two sets as follows:
- The **union** of two sets A and B, written \( A \cup B \), is the set of elements that are in A or in B (or both). (Again, in logic a similar symbol, \( \lor \), is used to connect two propositions with the OR operator.)
- So, for example, \( \{1,2,3,4\} \cup \{2,4,6,8\} = \{1,2,3,4,6,8\} \).

The \( \cup \) symbol looks like the first letter of ‘Union’ and like a cup that will hold a lot of items. The \( \cap \) symbol looks like a spilled cup that won’t hold a lot of items, or possibly the letter ‘n’ for intersection. Take care not to confuse the two.

**Difference**
- The **difference** of two sets A and B (also known as the set – **theoretic difference** of A and B, or the relative complement of B in A) is the set of elements that are in A but not in B.

This in written \( A - B \), or sometimes \( A \setminus B \).
- For example, if \( A = \{1,2,3,4\} \) and \( B = \{2,4,6,8\} \), then \( A - B = \{1,3\} \)

**Complement**
So far, we have considered operations in which two sets combine to form a third: binary operations. Now we look at a unitary operation – one that involves just one set.
- The set of elements that are not in a set A is called the **complement** of A. It is written \( A' \) (or sometimes \( A^c \)).

For example, if \( U=\mathbb{N} \) and \( A = \{\text{odd numbers}\} \), then \( A' = \{\text{even numbers}\} \)

**Cardinality**
Finally, in this section on set operations we look at an operation on a set that yields not another set, but an integer. The **cardinality** of a finite set A, written \( |A| \) (sometimes \#(A) or n(A)), is the number of (distinct) elements in A. So, for example:

If \( A = \{\text{lower case letters of the alphabet}\} \), \( |A| = 26 \).

When deciding how large finite sets are, we generally count the number of elements in the set, and say two sets are the same size if they have the same number of elements. This approach doesn’t work too well if the sets are infinite, however, because we can’t count the number of elements in an infinite set.

However, there is another way to define when two sets have the same size that works equally well for finite and infinite sets. We say that two sets A and B have the same size if we can define a function \( f : A \rightarrow B \) which satisfies the following properties:
- \( f(a) \) is defined for every \( a \in A \).
- \( \forall b \in B, \exists a \in A \) such that \( f(a) = b \). We say that f is onto or surjective.
• \( \forall a_1, a_2 \in A, f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \). We say that \( f \) is one-to-one or injective. Functions which satisfy these properties are called bisections.

**Power Sets**

The **power set** of a set \( A \) is the set of all its subsets (including, of course, itself and the empty set) It is denoted by \( \mathcal{P}(A) \).

Using set comprehension notation, \( \mathcal{P}(A) \) can be defined as:

\[
\mathcal{P}(A) = \{ Q \mid Q \subseteq A \}
\]

**The Foundational Rules of Set Theory**

The laws listed below can be described as the Foundational Rules of Set Theory. We derive them by going back to the definitions of intersection, union, universal set and empty set, and by considering whether a given element is in, or not in, one or more sets.

The Idempotent Laws: \( A \cap A = A \) and \( A \cup A = A \).

De Morgan’s Laws: \( (A \cup B)' = A' \cap B' \) and \( (A \cap B)' = A' \cup B' \).

Commutative Laws: \( A \cap B = B \cap A \) and \( A \cup B = B \cup A \).

Associative Laws: \( (A \cap (B \cap C) = A \cap (B \cap C) \) and \( (A \cup (B \cup C) = A \cup (B \cup C) \).

Distributive Laws: \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \) and \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \).

Involution Law: \( (A')' = A \)

**Cartesian Products**

**Ordered n–tuples**

In general, if we have \( n \) sets: \( A_1, A_2, \ldots, A_n \), then their Cartesian product is defined by:

\[
A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n \}
\]

And \( (a_1, a_2, \ldots, a_n) \) is called an **ordered n–tuple**.

**Complex Numbers**

**Complex numbers** are the extension of the real numbers, i.e., the number line, into a number plane. They allow us to turn the rules of plane geometry into arithmetic. Complex numbers have fundamental importance in describing the laws of the universe at the subatomic level, including the propagation of light and quantum mechanics. They also have practical uses in many fields, including signal processing and electrical engineering.

**Definition 1**

A **complex number** is an expression of the form \( x + iy \), in which \( x \) and \( y \) are real numbers and \( i \) is a new number, called the **imaginary unit**, for which expressions the normal rules of calculation apply together with the extra rule: \( i^2 = -1 \).

**Definition 2**

A **complex number** is a pair of real numbers \( (x, y) \), satisfying the properties:

\[
(a, b) + (c, d) = (a + c, b + d)
\]

\[
(a, b)(c, d) = (ac - bd, ad + bc)
\]

In both cases a complex number consists of two real numbers \( x \) and \( y \). The real number \( x \) is called the **real part** and the real number \( y \) is called the **imaginary part** of the complex number.
Conjugates
The conjugate of a complex number \( z \), written \( \bar{z} \), is the same number with the sign of the imaginary part changed: the conjugate of \( a + bi = a - bi \) (and vice versa).

Let us examine what happens when we have a complex number \( z = a + bi \), what is the product of \( z \) and its conjugate?

\[
(a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2
\]

Notice the imaginary parts cancel out, so the product is a real number. This will aid us greatly in the division of a complex number, as we will see.

Notice also that this is the sum of two squares, analogous to the difference of two squares.

The Argand Plane
We can represent complex numbers geometrically as well. Every complex number can be represented in the form \( z = x + iy \) (so \( x = \text{Re}(z) \) and \( y = \text{Im}(z) \)). Then we can represent \( z \) in the xy-plane by the point \((x,y)\). Notice that this is a one-to-one relationship: for each complex number, we have one corresponding point in the plane, and for each point in the plane corresponds one complex number. When we use the xy-plane in this way to represent complex numbers, we call the plane the "Argand plane". We will refer to the "y" axis as the imaginary axis, and the "x" axis as the real axis.

Modulus and Argument

\[
\text{On this diagram we can see the number } 3 + 4i. \text{ The line is the distance away from the origin (the number } 0 + 0i). \text{ We can see that the line makes an angle } \theta \text{ from the real axis.}
\]

It is clear that almost all complex numbers have this distance away from the origin and that almost all complex numbers make an angle away from the real axis. We give these two qualities special names; the distance away from the origin is known as the modulus of the complex number, and the angle \( \theta \) is known as the argument of the complex number.

We write the modulus of a complex number \( z \) by \( |z| \), and the argument of the complex number as \( \arg z \).

We can calculate the modulus and argument by basic trigonometry.

Calculating the modulus
In the above example, we have the number \( 3 + 4i \). We can form a triangle in the Argand plane with base 3 and height 4. By Pythagoras we can find the length of the hypotenuse by

\[
\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5
\]

and thus, the length of the hypotenuse is thus the modulus of the complex number, and it is 5 for \( 3 + 4i \).

Generalization
If \( z = x + yi \), \( |z| \) is clearly \( \sqrt{x^2 + y^2} \). Equivalently, \( |z| = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2} \).
Calculating the argument
We have the same triangle as we had in calculating the modulus. Remember from trigonometry that \( \tan \theta \) is the ratio of the height over base. So, for \( 3 + 4i \), we have \( \tan \theta = 4/3 \), and thus \( \theta = \arctan 4/3 = 0.9... \)
With complex numbers, we always take two things:
• the argument must be in radians.
• the argument lies in the interval \([-\pi, \pi]\), and we always adjust the angle so it does.
Note that \( \arg 0 \) is undefined.

Generalization
If \( z = x + yi \), \( \arg z \) is clearly \( \arctan(y/x) \), or, equivalently, \( \arg z = \arctan(\text{Im}(z) / \text{Re}(z)) \).

The polar form
We are now able to calculate the modulus and argument of a complex number, where these two numbers are able to uniquely describe every number in the Argand plane.
Using these two characteristics of complex numbers, we are now able to formulate a new way of writing these numbers.
Note that in the above diagram, we obtain a triangle that describes the complex number \( 3 + 4i \).
Clearly, we can do this for all complex numbers in the Argand plane (except for 0).
To simplify our work, let us look at numbers in the circle of unit length equidistant from 0. From trigonometry, we can parameterize all the points on a circle in the Cartesian plane by \( (\cos \theta, \sin \theta) \). In complex number notation we can say that all numbers on this unit circle are in the form \( \cos \theta + i \sin \theta \).
This works well on the unit circle, but how does this generalize to describing all numbers on the plane? We simply make the circle larger or smaller to encompass the number; this is done by multiplying by the modulus.
So then, we obtain the polar form \( r(\cos \theta + i \sin \theta) = z \), where \( r \) is the modulus.

Euler’s formula
A very significant result in the area of complex numbers is Euler’s formula. It basically asserts that \( re^{i\theta} = r(\cos \theta + i \sin \theta) \)
This statement can be verified through a rearrangement of the Taylor Series of the cosine and sine functions. Note that conjugate complex numbers have an opposing argument. \( 2e^{2i} \) and \( 2e^{-2i} \) are conjugate pairs.

De Moivre's theorem
De Moivre's theorem is useful in calculating powers of complex numbers. It states that \( (r(\cos \theta + i \sin \theta))^n = r^n (\cos n\theta + i \sin n\theta) \)
This follows clearly (from the laws of exponents) if we rewrite the theorem in the form \( (re^{i\theta})^n = r^n e^{in\theta} \)

Introduction to number theory
Unlike real analysis and calculus which deals with the dense set of real numbers, number theory examines mathematics in discrete sets, such as \( \mathbb{N} \) or \( \mathbb{Z} \).
Number Theory, the study of the integers, is one of the oldest and richest branches of mathematics. Its basic concepts are those of divisibility, prime numbers, and integer solutions to equations - all very simple to understand, but immediately giving rise to some of the best known theorems and