since $A^T y + s = c$.

The following theorem allows to check optimality of a primal and/or a dual solution.

Theorem 12 *(Complementary Slackness)*

Let x^*, (y^*, s^*) be feasible for (P), (D) respectively. The following are equivalent:

1. *x^* is an optimal solution to (P) and (y^*, s^*) is an optimal solution to (D).*

2. *$(s^*)^T x^* = 0$.*

3. *$x_j^* s_j^* = 0$, $\forall\, j = 1, \ldots, n$.*

4. *If $s_j^* > 0$ then $x_j^* = 0$.*

Proof:

Suppose (1) holds, then, by strong duality, $c^T x^* = b^T y^*$. Since $c = A^T y^* + s^*$ and $Ax^* = b$, we get that $(y^*)^T A x^* + (s^*)^T x^* = (x^*)^T A^T y^*$, and thus, $(s^*)^T x^* = 0$ (i.e (2) holds). It follows, since x_j^*, $s_j^* \geq 0$, that $x_j^* s_j^* = 0$, $\forall\, j = 1, \ldots, n$ (i.e. (3) holds). Hence, if $s_j^* > 0$ then $x_j^* = 0$, $\forall\, j = 1, \ldots, n$ (i.e. (4) holds). The converse also holds, and thus the proof is complete. □

In the example of section 9, the complementary slackness equations corresponding to the primal solution $x = (3, 2, 0)^T$ would be:

$$y_1 + 2y_2 = 1$$
$$y_1 + y_2 = 2$$

Note that this implies that $y_1 = 3$ and $y_2 = -1$. Since this solution satisfies the other constraint of the dual, y is dual feasible, proving that x is an optimum solution to the primal (and therefore y is an optimum solution to the dual).

11 Size of a Linear Program

11.1 Size of the Input

If we want to solve a Linear Program in polynomial time, we need to know what would that mean, i.e. what would the size of the input be. To this end we introduce two notions of the size of the input with respect to which the algorithm we present will run in polynomial time. The first measure of the input size will be the *size* of a *LP*, but we will introduce a new measure L of a *LP* that will be easier to work with. Moreover, we have that $L \leq \text{size}(LP)$, so that any algorithm running in time polynomial in L will also run in time polynomial in *size(LP)*.

Let's consider the linear program of the form:

$$\begin{aligned} &\min c^T x \\ s.t.& \\ &Ax = b \\ &x \geq 0 \end{aligned}$$

where we are given as inputs the coefficients of A (an $m \times n$ matrix), b (an $m \times 1$ vector), and c (an $n \times 1$ vector), whith rationial entries.

We can further assume, without loss of generality, that the given coefficients are all integers, since any LP with rational coefficients can be easily transformed into an equivalent one with integer coefficients (just multiply everything by l.c.d.). In the rest of these notes, we assume that A, b, c have integer coefficients.

For any integer n, we define its size as follows:

$$size(n) \triangleq 1 + \lceil \log_2(|n| + 1) \rceil$$

where the first 1 stands for the fact that we need one bit to store the sign of n, $size(n)$ represents the number of bits needed to encode n in binary. Analogously, we define the size of a $p \times 1$ vector d, and of a $p \times l$ matrix M as follows:

$$size(v) \triangleq \textstyle\sum_{i=1}^p size(v_i)$$
$$size(M) \triangleq \textstyle\sum_{i=1}^p \sum_{j=1}^l size(m_{ij})$$

We are then ready to talk about the size of a LP.

Definition 6 (Size of a linear program)

$$\text{size(LP)} \triangleq \text{size}(A) + \text{size}(b) + \text{size}(c).$$

A more convenient definition of the size of a linear program is given next.

Definition 7

$$L \triangleq \text{size}(det_{max}) + \text{size}(b_{max}) + \text{size}(c_{max}) + m + n$$

where

$$\begin{aligned} det_{max} &\triangleq \max_{A'}(|\det(A')|) \\ b_{max} &\triangleq \max_i(|b_i|) \\ c_{max} &\triangleq \max_j(|c_j|) \end{aligned}$$

and A' is any square submatrix of A.

Proposition 13 $L < \text{size(LP)},\ \forall A, b, c.$

Before proving this result, we first need the following lemma:

Lemma 14 *1. If $n \in \mathbb{Z}$ then $|n| \leq 2^{size(n)-1} - 1$.*

2. If $v \in \mathbb{Z}^n$ then $\|v\| \leq \|v\|_1 \leq 2^{size(v)-n} - 1$.

3. If $A \in \mathbb{Z}^{n \times n}$ then $|det(A)| \leq 2^{size(A)-n^2} - 1$.

Proof:

1. By definition.

2. $1 + \|v\| \leq 1 + \|v\|_1 = 1 + \sum_{i=1}^{n} |v_i| \leq \prod_{i=1}^{n}(1+|v_i|) \leq \prod_{i=1}^{n} 2^{size(v_i)-1} = 2^{size(v)-n}$ where we have used 1.

3. Let $a_1, \ldots, a_n$ be the columns of A. Since $|det(A)|$ represents the volume of the parallelepiped spanned by $a_1, \ldots, a_n$, we have

$$|det(A)| \leq \prod_{i=1}^{n} \|a_i\|.$$

Hence, by 2,

$$1 + |det(A)| \leq 1 + \prod_{i=1}^{n} \|a_i\| \leq \prod_{i=1}^{n}(1 + \|a_i\|) \leq \prod_{i=1}^{n} 2^{size(a_i)-n} = 2^{size(A)-n^2}.$$

□

We now prove Proposition 13.

Proof:

If B is a square submatrix of A then, by definition, $size(B) \leq size(A)$. Moreover, by lemma 14, $1 + |det(B)| \leq 2^{size(B)-1}$. Hence,

$$\lceil \log(1 + |det(B)|) \rceil \leq size(B) - 1 < size(B) \leq size(A). \tag{1}$$

Let $v \in \mathbb{Z}^p$. Then $size(v) \geq size(\max_j |v_j|) + p - 1 = \lceil \log(1 + \max_j |v_j|) \rceil + p$. Hence,

$$size(b) + size(c) \geq \lceil \log(1 + \max_j |c_j|) \rceil + \lceil \log(1 + \max_i |b_i|) \rceil + m + n. \tag{2}$$

Combining equations (1) and (2), we obtain the desired result. □

Remark 1 *$det_{max} * b_{max} * c_{max} * 2^{m+n} < 2^L$, since for any integer n, $2^{\text{size}(n)} > |n|$.*

In what follows we will work with L as the size of the input to our algorithm.

11.2 Size of the Output

In order to even hope to solve a linear program in polynomial time, we better make sure that the solution is representable in size polynomial in L. We know already that if the LP is feasible, there is at least one vertex which is an optimal solution. Thus, when finding an optimal solution to the LP, it makes sense to restrict our attention to vertices only. The following theorem makes sure that vertices have a compact representation.

Theorem 15 *Let x be a vertex of the polyhedron defined by $Ax = b, x \geq 0$. Then,*

$$x^T = \left(\frac{p_1}{q} \; \frac{p_2}{q} \ldots \frac{p_n}{q} \right),$$
$$\text{where } p_i \; (i = 1, \ldots, n), \; q \in \mathbb{N},$$

and

$$0 \leq p_i < 2^L$$
$$1 \leq q < 2^L.$$

Proof:

Since x is a basic feasible solution, $\exists$ a basis B such that $x_B = A_B^{-1} b$ and $x_N = 0$. Thus, we can set $p_j = 0$, $\forall \; j \in N$, and focus our attention on the x_j's such that $j \in B$. We know by linear algebra that

$$x_B = A_B^{-1} b = \frac{1}{det(A_B)} cof(A_B) b$$

where $cof(A_B)$ is the cofactor matrix of A_B. Every entry of A_B consists of a determinant of some submatrix of A. Let $q = |det(A_B)|$, then q is an integer since A_B has integer components, $q \geq 1$ since A_B is invertible, and $q \leq det_{\max} < 2^L$. Finally, note that $p_B = q x_B = |cof(A_B) b|$, thus $p_i \leq \sum_{j=1}^{m} |cof(A_B)_{ij}||b_j| \leq m \; det_{\max} \; b_{\max} < 2^L$.
□

12 Complexity of linear programming

In this section, we show that linear programming is in NP$\cap$ co-NP. This will follow from duality and the estimates on the size of any vertex given in the previous section. Let us define the following decision problem:

Definition 8 ($\mathcal{LP}$)

<u>*Input:*</u> *Integral A, b, c,* and a rational number λ*,*
<u>*Question:*</u> Is $\min\{c^T x : Ax = b, \; x \geq 0\} \leq \lambda$*?*