0-1 Knapsack Inequalities

- Valid Inequalities for the Knapsack Problem. We are interested in valid inequalities for the knapsack set

\[ S = \{ x \in \{0, 1\}^n \mid \sum_{j=1}^{N} a_j x_j \leq b \}. \]

\( N = \{1, 2, \ldots, n\} \). Assume that \( a_j > 0, j \in N, a_j < b, j \in N \). We are interested in finding facets of \( \text{conv}(S) \).

- Simple facets. What is \( \text{dim}(\text{conv}(S)) \)? \( 0, e_j, j \in N \) are \( n + 1 \) affinely independent points in \( \text{conv}(S) \), so \( \text{dim}(\text{conv}(S)) = n \).
- \( x^k \geq 0 \) is a facet of \( \text{conv}(S) \).
- Proof. \( 0, e_j, j \in N \setminus \{k\} \) are \( n \) affinely independent points that satisfy \( x_k = 0 \).
- $x_k \leq 1$ is a facet of $\text{conv}(S)$ if $a_j + a_k \leq b$, $\forall j \in N \setminus \{k\}$.

- Proof. $e_k, e_j + e_k, j \in N \setminus \{k\}$ are $n$ affinely independent points that satisfy $x_k = 1$.

- A set $C \subseteq N$ is a cover if $\sum_{j \in C} a_j > b$. A cover $C$ is a minimal cover if $C \setminus \{j\}$ is not a cover $\forall j \in C$.

- If $C \subseteq N$ is a cover, then the cover inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is a valid inequality for $S$. 


Example:

\[ S = \{ x \in B^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \}. \]

Minimal covers:

\[ C = \{1, 2, 3\}, C = \{1, 2, 6\}, \]
\[ C = \{1, 5, 6\}, C = \{3, 4, 5, 6\}. \]
Can we do better?

- Are these inequalities the strongest ones we can come up with? What does strongest mean? We all know that facets are the “strongest”, but can we say anything else?
- If $\pi x \leq \pi_0$ and $\mu x \leq \mu_0$ are two valid inequalities for $P \in \mathbb{R}_+^n$, we say that $\pi x \leq \pi_0$ dominates $\pi x \leq \mu_0$ if $\exists u \geq 0$ such that $\pi \geq u\mu$, $\pi_0 \leq u\mu_0$ and $(\pi, \pi_0) \neq u(\mu, \mu_0)$.
- If $\pi x \leq \pi_0$ dominates $\pi x \leq \mu_0$, then
  $$\{x \in \mathbb{R}_+^n \mid \pi x \leq \pi_0\} \subseteq \{x \in \mathbb{R}_+^n \mid \mu x \leq \mu_0\}.$$
Strengthening cover inequalities. If $C \subseteq N$ is a minimal cover, the extended cover $E(C)$ is defined as $E(C) = C \cup \{j \in N \mid a_j \geq a_i, \forall i \in C\}$. If $E(C)$ is an extended cover for $S$, then the extended cover inequality

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is a valid inequality for $S$.

The cover inequality $x_3 + x_4 + x_5 + x_6 \leq 3$ is dominated by the extended cover inequality $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$. 
Let $C$ be a minimal cover. If $C = N$, then
\[ \sum_{j \in C} x_j \leq |C| - 1 \]
is a facet of $\text{conv}(S)$.

Proof $R_k = C \setminus \{k\}, \forall k \in C$. $x^{R_k}$ satisfies
\[ \sum_{j \in C} x_j^{R_k} = |C| - 1. \]
Also, $x^{R_1}, \ldots, x^{R_{|C|}}$ are affinely independent. Since $C = N$, there are $n$ affinely independent vectors satisfy
\[ \sum_{j \in C} x_j^{R_k} = |C| - 1 \] at equality.
Let $C = \{j_1, \ldots, j_r\}$ be a minimal cover. Let $p = \min\{j \mid j \in N \setminus E(C)\}$. If $C = E(C)$, and $\sum_{j \in C \setminus \{j_1\}} a_j + a_p \leq b$, then $\sum_{j \in C} x_j \leq |C| - 1$ is a facet of $\text{conv}(S)$.

**Proof.** $T_k = C \setminus \{j_1\} \cup \{k\}, \forall k \in N \setminus E(C)$. $|T_k \cap E(C)| = |C| - 1$ and

$$\sum_{j \in T_k \cap E(C)} x_j^{T_k} = |C| - 1.$$

$|R_k| + |T_k| = N$. $x^{R_k}$, $k \in C$, $x^{T_k}$, $k \in N \setminus C$, are $n$ affinely independent vectors.
Example:

\[ S = \sum_{x \in \{0, 1\}^5} (79x_1 + 53x_2 + 53x_3 + 45x_4 + 45x_5 \leq 178). \]

Consider minimal cover \( C = \{1, 2, 3\} \). The valid inequality is:

\[ x_1 + x_2 + x_3 \leq 2. \]

\( C = E(C) \). \( p = 4 \), \( C \setminus \{1\} \cup \{4\} = \{2, 3, 4\} \).

\( 53 + 53 + 45 = 151 < 178 \). So \( x_1 + x_2 + x_3 = 2 \) gives a facet of \( \text{conv}(S) \).
Lifting Cover Inequalities

- **Question:** Can we find the valid inequality as strong as possible?
- **Example:** $C = \{3, 4, 5, 6\}$, the valid inequality for $C$ is:
  \[ x_3 + x_4 + x_5 + x_6 \leq 3. \]
- Setting $x_1 = x_2 = x_7 = 0$, the cover inequalities $x_3 + x_4 + x_5 + x_6 \leq 3$ is valid for
  \( \{x \in \{0, 1\}^4 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19\} \).
- If $x_1$ is not fixed at 0, can we strengthen the inequality? For what values of $\alpha_1$ is the inequality
  \[ \alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3 \]
  valid for
  \( P_{2,7} = \{x \in \{0, 1\}^5 \mid 11x_1 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19\} \).
\(\Leftrightarrow \alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3\) is valid for all \(x \in \{0, 1\}^4\)
satisfying \(6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 11;\)
\(\Leftrightarrow \alpha_1 + \zeta \leq 3,\) where
\[
\zeta = \max\{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\}.
\]
\(\Rightarrow \zeta = 1 \Rightarrow \alpha_1 \leq 2.\) Thus \(\alpha_1 = 2\) gives the strongest inequality.

\(\Rightarrow\) How to find the best value \(\alpha_j, j \in N \setminus C\) such that
\[
\sum_{j \in C} x_j + \sum_{j \in N \setminus C} \alpha_j x_j \leq |C| - 1
\]
is valid for \(S\)?
Lifting Procedure

Let $j_1, \ldots, j_r$ be an ordering of $N \setminus C$. Set $t = 1$.

The valid inequality

$$\sum_{i=1}^{t-1} \alpha_{i}x_{j_i} + \sum_{j \in C} x_j \leq |C| - 1$$

is given. Solve the following knapsack problem:

$$\zeta_t = \max_{x} \sum_{i=1}^{t-1} \alpha_{i}x_{j_i} + \sum_{j \in C} x_j$$

subject to

$$\sum_{i=1}^{t-1} a_{i}x_{j_i} + \sum_{j \in C} a_jx_j \leq b - a_{j_t}$$

$$x \in \{0, 1\}^{|C|+t-1}.$$

Set $\alpha_{j_t} = |C| - 1 - \zeta_t$. Stop if $t = r$. 
Example: \( C = \{3, 4, 5, 6\}, j_1 = 1, j_2 = 2, j_3 = 7. \) \( \alpha_3 = 2. \)

Consider \( x_3 \), we have

\[
\zeta_2 = \max_{\mathbf{x}} 2x_1 + x_3 + x_4 + x_5 + x_6 \\
\text{s.t.} \quad 11x_1 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 6 = 13, \\
\mathbf{x} \in \{0, 1\}^5.
\]

So \( \zeta_2 = 2 \) and \( \alpha_{j_2} = \alpha_2 = 3 - 2 = 1. \)

Consider \( x_7 \) now, we have

\[
\zeta_3 = \max_{\mathbf{x}} 2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
\text{s.t.} \quad 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 1 = 18, \\
\mathbf{x} \in \{0, 1\}^6.
\]

So \( \zeta_3 = 3 \) and \( \alpha_{j_3} = \alpha_7 = 3 - 3 = 0. \) We obtain a valid inequality:

\[
2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 3.
\]
Separation of Cover Inequalities

- We often try to solve problems that have knapsack rows with lots more variables than that... Obviously I do not want to add all of those facets at once. **What to do?**

- Given some $P$, find an inequality of the form
  \[ \sum_{j \in C} x_j \leq |C| - 1 \] such that \[ \sum_{j \in C} x_j^* > |C| - 1 \]. This is called a **separation problem**.

- Note that \[ \sum_{j \in C} x_j \leq |C| - 1 \] can be rewritten as
  \[ \sum_{j \in C} (1 - x_j) \geq 1. \]
Separation Problem: Given a fractional LP solution $x^*$, does there exist a cover $C \subseteq N$ such that $\sum_{j \in C} (1 - x^*_j) < 1$? or is

$$\gamma = \min_{C \subseteq N} \left\{ \sum_{j \in C} (1 - x_j) \mid \sum_{j \in C} a_j > b \right\} < 1?$$

- If $\gamma \geq 1$, then $x^*$ satisfies all the cover inequalities.
- If $\gamma < 1$ with optimal solution $z^R$, then $\sum_{j \in R} x_j \leq |R| - 1$ is a violated cover inequality.