1. Preprocessing

LP or IP models can often be simplified by reducing the number of variables and constraints (e.g. eliminating the redundant constraints), and IP models can be tightened before any actual branch- and-bound computations are performed. All the commercial branch-bound systems carry out such a check, called preprocessing.

Example 13.1.

Consider the LP instance

\[
\begin{align*}
\text{max } & \ 2x_1 + x_2 - x_3 \\
\text{s.t. } & \ 5x_1 - 2x_2 + 8x_3 \leq 15, \\
& \ 8x_1 + 3x_2 - x_3 \geq 9, \\
& \ x_1 + x_2 + x_3 \leq 6, \\
& \ 0 \leq x_1 \leq 3, \\
& \ 0 \leq x_2 \leq 1, \\
& \ 1 \leq x_3.
\end{align*}
\]

1.1 Tightening bounds:
- Isolating $x_1$ in the first constraint and using $x_2 \leq 1$, $x_3 \geq 1$ yields

$$5x_1 \leq 15 + 2x_2 - 8x_3 \leq 15 + 2 \times 1 - 8 \times 1 = 9,$$

and hence, $x_1 \leq 9/5$, which tightens the bound $x_1 \leq 3$.

- Similarly, isolating $x_3$ in the first constraint, we obtain

$$8x_3 \leq 15 + 2x_2 - 5x_1 \leq 15 + 2 \times 1 - 5 \times 0 = 17.$$

which implies $x_3 \leq 17/8$ and it tightens $x_3 \leq \infty$.

- Isolating $x_2$ in the first constraint,

$$2x_2 \geq 5x_1 + 8x_3 - 15 \geq 5 \times 0 + 8 \times 1 - 15 = -7.$$

yields $x_2 \geq -7/2$ which does not tighten $x_2 \geq 0$.

- Proceeding similarly with the second and third constraints, we obtain the tightened bound

$$8x_1 \geq 9 - 3x_2 + x_3 \geq 9 - 3 + 1 = 7,$$

yielding the improved bound $x_1 \geq 7/8$. 
As some of the bounds have changed after the first sweep, we may now go back to the first constraint and tighten the bounds yet further. Isolating $x_3$, we obtain
\[
8x_3 \leq 15 + 2x_2 - 5x_1 \leq 15 + 2 - 5 \times \frac{7}{8} = 101/8
\]
yielding the improved bound $x_3 \leq 101/64$.

Continuing the second sweep by isolating each variable in turn in each of the constraints 1-3, and using the bound constraints, several bound constraints may further tighten in general, but not in the present example.

How many sweeps of this process are needed? One can show that after two sweeps of all the constraints and variables, the bounds cannot improve further!
1.2 Redundant Constraints:

Using the final upper bounds in constraint 3,

\[ x_1 + x_2 + x_3 \leq \frac{9}{5} + 1 + \frac{101}{64} < 6 \]

so that this constraint is redundant and can be omitted.

The remaining problem is

\[
\begin{align*}
\text{max } & \quad 2x_1 + x_2 - x_3 \\
& \quad 5x_1 - 2x_2 + 8x_3 \leq 15, \\
& \quad 8x_1 + 3x_2 - x_3 \geq 9, \\
& \quad \frac{7}{8} \leq x_1 \leq \frac{9}{5}, \quad 0 \leq x_2 \leq 1, \quad 1 \leq x_3 \leq \frac{101}{64}.
\end{align*}
\]
1.3 Variable fixing:

- Increasing $x_2$ makes the objective function grow and loosens all constraints except $x_2 \leq 1$. Therefore, in an optimal solution we must have $x_2 = 1$.

- Decreasing $x_3$ makes the objective function grow and loosens all constraints except $1 \leq x_3$. Thus, in an optimal solution we must have $x_3 = 1$.

This leaves the trivial problem

$$\max \{2x_1 : 7/8 \leq x_1 \leq 9/5\}.$$  

Example 13.1 shows how to simplify linear programming instances. In the preprocessing of IPs we have further possibilities:

1. For all $x_j$ with an integrality constraint $x_j \in Z$ any bounds $l_j \leq x_j \leq u_j$ can be tightened to $[l_j] \leq x_j \leq [u_j]$. 
We formalize the ideas from our example above:

Observation 8.6 Consider the set

\[ S = \left\{ x : a_0 x_0 + \sum_{j=1}^{n} a_j x_j \leq b, \ l_j \leq x_j \leq u_j, \ for \ j = 1, \ldots, n \right\}. \]

The following statements hold:

Bounds on variables If \( a_0 > 0 \), then

\[ x_0 \leq \left( b - \sum_{j:a_j>0} a_j l_j - \sum_{j:a_j<0} a_j u_j \right) / a_0. \]

and, if \( a_0 < 0 \), then

\[ x_0 \geq \left( b - \sum_{j:a_j>0} a_j l_j - \sum_{j:a_j<0} a_j u_j \right) / a_0. \]

Redundancy The constraint \( a_0 x_0 + \sum_{j=1}^{n} a_j x_j \leq b \) is redundant, if

\[ \sum_{j:a_j>0} a_j u_j + \sum_{j:a_j<0} a_j l_j \leq b. \]

Infeasibility The set \( S \) is empty, if

\[ \sum_{j:a_j>0} a_j l_j + \sum_{j:a_j<0} a_j u_j > b. \]

Variable Fixing For a maximization problem of the form \( \max \{ c^T x : Ax \leq b, \ l \leq x \leq u \} \),

if \( a_{ij} \geq 0 \) for all \( i = 1, \ldots, m \) and \( c_j < 0 \), then \( x_j = l_j \) in an optimal solution.

Conversely, if \( a_{ij} \leq 0 \) for all \( i = 1, \ldots, m \) and \( c_j > 0 \), then \( x_j = u_j \) in an optimal solution.
Example 13.2.

Consider a binary IP instance whose feasible set is defined by the following constraints:

\[
\begin{align*}
7x_1 + 3x_2 - 4x_3 - 2x_4 & \leq 1, \\
-2x_1 + 7x_2 + 3x_3 + x_4 & \leq 6, \\
-2x_2 - 3x_3 - 6x_4 & \leq -5, \\
3x_1 - 2x_3 & \geq -1, \\
x & \in \mathbb{B}^4.
\end{align*}
\]

The latter point is illustrated in the next example.
1.4 Generating logical inequalities:

The first constraint shows that \( x_1 = 1 \implies x_3 = 1 \), which can be written as

\[ x_1 \leq x_3. \]

Likewise, \( x_1 = 1 \implies x_4 = 1 \), or equivalently,

\[ x_1 \leq x_4. \]

Finally, constraint 1 also shows that the problem is infeasible if \( x_1 = x_2 = 1 \). Therefore, the following constraint must hold,

\[ x_1 + x_2 \leq 1. \]

We can process the remaining constraints in a similar way:

A) Constraint 2 yields the inequalities \( x_2 \leq x_1 \) and \( x_2 + x_3 \leq 1 \).

B) Constraint 3 yields \( x_2 + x_4 \geq 1 \) and \( x_3 + x_4 \geq 1 \).

C) Constraint 4 yields \( x_1 \geq x_3 \).
Although the introduction of the new logical constraints makes the problem seem more complicated, the formulation becomes tighter and thus better. Furthermore, we can now process the problem further:

1.5 Combining pairs of logical inequalities:

We now consider pairs involving the same variables.

\[ x_1 \leq x_3 \text{ and } x_1 \geq x_3 \implies x_1 = x_3. \]

\[ x_1 + x_2 \leq 1 \text{ and } x_2 \leq x_1 \implies x_2 = 0, \]

and then

\[ x_2 + x_4 \geq 1 \implies x_4 = 1. \]

1.6 Simplifying:

Substituting the identities \( x_2 = 0, x_3 = x_1 \) and \( x_4 = 1 \) we found, all four constraints become redundant. We are left with the choice \( x_1 \in \{0,1\} \), and hence the feasible set contains only two points

\[ S = \{(1,0,1,1), (0,0,0,1)\}. \]