Self-duality

A self-dual linear program has the following general form:

$$\min_{u,w} f^T u + g^T w$$

$$\begin{array}{rcl} M_{11}u + M_{12}w & \geq & -f, \\ -M_{12}^T u + M_{22}w & = & -g, \end{array} \qquad u \geq 0,\ w \text{ free}, \qquad \text{(SDP)}$$

where the matrices M_{11} and M_{22} are square and skew-symmetric, that is, $M_{11} = -M_{11}^T$ and $M_{22} = -M_{22}^T$.

- The vector (u, w) is a solution of SDP iff vector (u, v, w) is a solution of the following mixed monotone linear complementarity problem(mLCP):

$$\begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ -M_{12}^T & M_{22} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix}, \quad (u, v) \geq 0, \quad u^T v = 0.$$

Self-duality

By applying the K.K.T conditions , we find that a solution (u, w) of SDP must satisfy

$$\begin{aligned} M_{11}\bar{u} + M_{12}\bar{w} &\geq -f, \\ -M_{12}^T\bar{u} + M_{22}\bar{w} &= -g, \\ M_{11}u + M_{12}w &\geq -f, \\ -M_{12}^T u + M_{22}w &= -g, \\ u^T\left[M_{11}\bar{u} + M_{12}\bar{w} + f\right] &= 0, \\ \bar{u}^T\left[M_{11}u + M_{12}w + f\right] &= 0, \\ u \geq 0,\ w \text{ free},\ \bar{u} \geq 0,\ \bar{w} \text{ free}, \end{aligned}$$

where $(\bar{u}, \bar{w})$ is a vector of Lagrange multipliers. Noting that the problem is self-dual, therefore any solution of the primal is a solution of the dual. So we can set $(\bar{u}, \bar{w}) = (u, w)$ and eliminate the redundancies to obtain

Self-duality

$$\begin{aligned} M_{11}u + M_{12}w &\geq -f, \\ -M_{12}^T u + M_{22}w &= -g, \\ u^T [M_{11}u + M_{12}w + f] &= 0, \\ u \geq 0,\ w \text{ free.} \end{aligned}$$

By introducing the variable v, we obtain the following mixed monotone linear complementarity (mLCP) formulation :

$$\begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ -M_{12}^T & M_{22} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix}, \quad (u, v) \geq 0, \quad u^T v = 0.$$

By construction of the mLCP , the vector $(u, v, w) = (u, M_{11}u + M_{12}w + f, w)$ is solution of mLCP whenever (u, w) is a solution of SDP. If (u, v, w) solves mLCP, then (u, w) solves SDP. It is easily verified by setting the Lagrange multipliers in the K.K.T conditions to $(\bar{u}, \bar{w}) = (u, w)$.

Self-duality

- Equivalence between mLCP and K.K.T conditions is the key to efficient solution of self-dual problem. If we were to apply a primal-dual method to these problems, the unknowns would be $u,\ w,\ \bar{u},\ \bar{w}$, together with the slack variables v and $\bar{v}$. But because of equivalence with mLCP , it is enough to solve a constrained nonlinear system with half as many variables , namely

$$F_{\text{mLCP}}(u,v,w) = \begin{bmatrix} f + M_{11}u + M_{12}w - v \\ g - M_{12}^T u + M_{22}w \\ UVe \end{bmatrix} = 0, \quad (u,v) \geq 0,$$

where $U = \text{diag}(u_1, u_2, \ldots)$ and $V = \text{diag}(v_1, v_2, \ldots)$.

Simplified Hsd form

Theorem. If the self-dual linear program is feasible, the LCP formulation has a strictly complementarity solution, that is , a solution with $u + v > 0$.

- Now consider the following primal-dual problems:

$$\begin{aligned} &\min c^T x \\ &\quad Ax = b, \\ &\quad x \geq 0, \end{aligned} \quad \xrightarrow{\textbf{dual}} \quad \begin{aligned} &\max b^T \lambda \\ &\quad A^T \lambda + s = c, \\ &\quad s \geq 0, \end{aligned} \qquad (6)$$

Simplified Hsd form

We now introduce a scalar variable τ and define the simplified HSD linear program as:

$$\begin{array}{llllll} \min & & & 0 & & \\ \text{subject to} & & -c^T x & +b^T \lambda & \geq & 0, \\ & c\tau & & -A^T \lambda & \geq & 0, \\ & -b\tau & +Ax & & = & 0, \\ & \lambda \text{ free}, (\tau, x) \geq 0. & & & & \end{array} \qquad \text{(SHSD)}$$

- The above problem is self-dual . It is homogeneous, sine the right-hand sides of all the constraints are zero.

$$M_{11} = \begin{bmatrix} 0 & -c^T \\ c & 0 \end{bmatrix}, \ M_{12} = \begin{bmatrix} b^T \\ -A^T \end{bmatrix}, \ M_{22} = 0,$$

$$f = g = 0, \quad u = (\tau, x), \ w = y.$$

Simplified Hsd form

- The vector (τ, x, λ) is a solution of SHSD iff $(\tau, x, \kappa, s, \lambda)$ is a solution of the following mLCP :

$$\begin{bmatrix} \kappa \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -c^T & b^T \\ c & 0 & -A^T \\ -b & A & 0 \end{bmatrix} \begin{bmatrix} \tau \\ x \\ \lambda \end{bmatrix}, \qquad \text{(mLCP)}$$

$$(\tau, x, \kappa, s) \geq 0, \qquad \tau\kappa + x^T s = 0,$$

- $(\tau^*, x^*, \lambda^*, \kappa^*, s^*)$ is a strictly complementary solution of mLCP if

$$\tau^* + \kappa^* > 0, \qquad x^* + s^* > 0.$$

- Since the SHSD problem is feasible, mLCP has a strictly complementary solution.

Simplified Hsd form

- The SHSD problem is feasible, but does not any strictly feasible solution. Let $(\bar{\tau}, \bar{x}, \bar{\lambda})$ be a strictly feasible solution of SHSD. Then

$$b^T \bar{\lambda} > c^T \bar{x}, \qquad A^T \bar{\lambda} < \bar{\tau} c, \qquad A\bar{x} = b, \qquad (\bar{\tau}, \bar{x}) > 0.$$

Hence, then point $(x, \lambda, s) = (\bar{x}/\bar{\tau}, \bar{\lambda}/\bar{\tau}, c - A^T \bar{\lambda}/\bar{\tau})$ is strictly feasible primal-dual point for (6). Therefore we obtain:

$$0 < x^T s = c^T x - b^T \lambda = \frac{1}{\bar{\tau}}(c^T \bar{x} - b^T \bar{\lambda}) < 0,$$

- Therefore SHSD can not be solved using feasible interior point method. However one may use infeasible interior point method.

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Simplified Hsd form

Theorem. The primal-dual solution set Ω for (6) is nonempty iff all strictly complementary solutions $(\tau^*, x^*, \lambda^*, \kappa^*, s^*)$ of mLCP have $\tau^* > 0 \ and \ \kappa^* = 0.$

Lemma. Let $(\tau^*, x^*, \lambda^*, \kappa^*, s^*)$ be a strictly complementary solution of mLCP with $\tau^* > 0 \ and \ \kappa^* = 0$. Then $(x, \lambda, s) = \frac{1}{\tau^*}(x^*, \lambda^*, s^*)$ is a primal-dual solution of (6).

Theorem. Suppose that mLCP has a strictly complementary solution $(\tau^*, x^*, \lambda^*, \kappa^*, s^*)$ for which $\kappa^* > 0$. then at least one of $c^T x^*$ and $-b^T \lambda^*$ is negative and:

- if $c^T \underline{x}^* < 0$, the dual problem is infeasible.
- If $-b^T \lambda^* < 0$, the primal problem is infeasible.

HSD FORM

Given initial points for the primal-dual vector (x^0, λ^0, s^0) with $x^0 > 0$ and $s^0 > 0$, the HSD for is define as:

$$
\begin{array}{llllllll}
\min & & & & ((x^0)^T s^0 + 1)\theta & & \\
\text{subject to} & & -c^T x & +b^T\lambda & +\bar{z}\theta & \geq & 0, \\
& c\tau & & -A^T\lambda & -\bar{c}\theta & \geq & 0, \\
& -b\tau & +Ax & & +\bar{b}\theta & = & 0, \\
& -\bar{z}\tau & +\bar{c}^T x & -\bar{b}^T\lambda & & = & -((x^0)^T s^0 + 1), \\
& & (\lambda, \theta) \text{ free}, & (\tau, x) \geq 0, & & &
\end{array}
$$

where

$$\bar{b} = b - Ax^0, \qquad \bar{c} = c - A^T\lambda^0 - s^0, \qquad \bar{z} = c^T x^0 + 1 - b^T\lambda^0.$$

HSD FORM

- HSD problem is self-dual.
- For the equivalent mLCP formulation for HSD , we introduce slack varaibles $\kappa \in \mathbb{R}$ and $s \in \mathbb{R}^n$ for the first and second constraints, respectively. So we obtain:

$$\begin{bmatrix} \kappa \\ s \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -c^T & b^T & \bar{z} \\ c & 0 & -A^T & -\bar{c} \\ -b & A & 0 & \bar{b} \\ -\bar{z} & \bar{c}^T & -\bar{b}^T & 0 \end{bmatrix} \begin{bmatrix} \tau \\ x \\ \lambda \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ (x^0)^T s^0 + 1 \end{bmatrix}, \qquad (7)$$

$$(\tau, x, \kappa, s) \geq 0, \qquad \tau\kappa + x^T s = 0.$$

- $(\tau, x, \lambda, \theta, \kappa, s) = (1, x^0, \lambda^0, 1, 1, s^0)$. is a strictly feasible solution for above mLCP problem and $(\tau, x, \lambda, \theta) = (1, x^0, \lambda^0, 1)$ is a strictly feasible solution for HSD.