Self-duality

A self-dual linear program has the following general form:

\[
\begin{align*}
\min_{u,w} & \quad f^T u + g^T w \\
M_{11} u + M_{12} w & \geq -f, \\
-M_{12}^T u + M_{22} w & = -g,
\end{align*}
\]

\[u \geq 0, \ w \ \text{free},\]

(SDP)

where the matrices \(M_{11}\) and \(M_{22}\) are square and skew-symmetric, that is, \(M_{11} = -M_{11}^T\) and \(M_{22} = -M_{22}^T\).

- The vector \((u, w)\) is a solution of SDP iff vector \((u, v, w)\) is a solution of the following mixed monotone linear complementarity problem (mLCP):

\[
\begin{bmatrix}
v \\
0
\end{bmatrix} = \begin{bmatrix}
M_{11} & M_{12} \\
-M_{12}^T & M_{22}
\end{bmatrix} \begin{bmatrix}
u \\
w
\end{bmatrix} + \begin{bmatrix}
f \\
g
\end{bmatrix}, \quad (u, v) \geq 0, \quad u^Tv = 0.
\]
By applying the K.K.T conditions, we find that a solution \((u, w)\) of SDP must satisfy

\[
M_{11}\overline{u} + M_{12}\overline{w} \geq -f, \\
-M_{12}^T\overline{u} + M_{22}\overline{w} = -g, \\
M_{11}u + M_{12}w \geq -f, \\
-M_{12}^Tu + M_{22}w = -g, \\
u^T [M_{11}\overline{u} + M_{12}\overline{w} + f] = 0, \\
\overline{u}^T [M_{11}u + M_{12}w + f] = 0, \\
u \geq 0, \ w \text{ free, } \overline{u} \geq 0, \ \overline{w} \text{ free},
\]

where \((\overline{u}, \overline{w})\) is a vector of Lagrange multipliers. Noting that the problem is self-dual, therefore any solution of the primal is a solution of the dual. So we can set \((\overline{u}, \overline{w}) = (u, w)\) and eliminate the redundancies to obtain
Self-duality

\[ M_{11} u + M_{12} w \geq -f, \]
\[ -M_{12}^T u + M_{22} w = -g, \]
\[ u^T [M_{11} u + M_{12} w + f] = 0, \]
\[ u \geq 0, \; w \text{ free}. \]

By introducing the variable \( v \), we obtain the following mixed monotone linear complementarity (mLCP) formulation:

\[
\begin{bmatrix}
    v \\
    0
\end{bmatrix} =
\begin{bmatrix}
    M_{11} & M_{12} \\
    -M_{12}^T & M_{22}
\end{bmatrix}
\begin{bmatrix}
    u \\
    w
\end{bmatrix} +
\begin{bmatrix}
    f \\
    g
\end{bmatrix}, \quad (u, v) \geq 0, \quad u^T v = 0.
\]

By construction of the mLCP, the vector \((u, v, w) = (u, M_{11} u + M_{12} w + f, w)\) is solution of mLCP whenever \((u, w)\) is a solution of SDP. If \((u, v, w)\) solves mLCP, then \((u, w)\) solves SDP. It is easily verified by setting the Lagrange multipliers in the K.K.T conditions to \((\bar{u}, \bar{w}) = (u, w)\).
**Self-duality**

- Equivalence between mLCP and K.K.T conditions is the key to efficient solution of self-dual problem. If we were to apply a primal-dual method to these problems, the unknowns would be $u, w, \bar{u}, \bar{w}$, together with the slack variables $v$ and $\bar{v}$. But because of equivalence with mLCP, it is enough to solve a constrained nonlinear system with half as many variables, namely

$$F_{mLCP}(u, v, w) = \begin{bmatrix} f + M_{11}u + M_{12}w - v \\ g - M_{12}^Tu + M_{22}w \\ UVe \end{bmatrix} = 0, \quad (u, v) \geq 0,$$

where $U = \text{diag}(u_1, u_2, \ldots)$ and $V = \text{diag}(v_1, v_2, \ldots)$. 
Theorem. If the self-dual linear program is feasible, the LCP formulation has a strictly complementarity solution, that is, a solution with $u + v > 0$.

- Now consider the following primal-dual problems:

$$
\begin{align*}
\min & \quad c^T x \\
Ax &= b, \\
x &\geq 0,
\end{align*}
\quad \text{dual}
\begin{align*}
\max & \quad b^T \lambda \\
A^T \lambda + s &= c, \\
s &\geq 0,
\end{align*}
$$

(6)
We now introduce a scalar variable $\tau$ and define the simplified HSD linear program as:

$$\min \quad 0$$

subject to

$$-c^T x + b^T \lambda \geq 0,$$

$$c \tau - A^T \lambda \geq 0,$$

$$-b \tau + Ax = 0,$$

$\lambda$ free, $(\tau, x) \geq 0.$

The above problem is self-dual. It is homogeneous, since the right-hand sides of all the constraints are zero.

$$M_{11} = \begin{bmatrix} 0 & -c^T \\ c & 0 \end{bmatrix}, \quad M_{12} = \begin{bmatrix} b^T \\ -A^T \end{bmatrix}, \quad M_{22} = 0,$$

$$f = g = 0, \quad u = (\tau, x), \quad w = y.$$
Simplified Hsd form

• The vector \((\tau, x, \lambda)\) is a solution of SHSD \(\text{iff } (\tau, x, \kappa, s, \lambda)\) is a solution of the following mLCP:

\[
\begin{bmatrix}
\kappa \\
0
\end{bmatrix} = \begin{bmatrix}
0 & -c^T & b^T \\
c & 0 & -A^T \\
-b & A & 0
\end{bmatrix} \begin{bmatrix}
\tau \\
x \\
\lambda
\end{bmatrix}, \quad \text{(mLCP)}
\]

\((\tau, x, \kappa, s) \geq 0, \quad \tau \kappa + x^T s = 0,\)

• \((\tau^*, x^*, \lambda^*, \kappa^*, s^*)\) is a strictly complementary solution of mLCP if \(\tau^* + \kappa^* > 0, \quad x^* + s^* > 0.\)

• Since the SHSD problem is feasible, mLCP has a strictly complementary solution.
Simplified Hsd form

- The SHSD problem is feasible, but does not any strictly feasible solution. Let $(\bar{\tau}, \bar{x}, \bar{\lambda})$ be a strictly feasible solution of SHSD. Then

\[
b^T \bar{\lambda} > c^T \bar{x}, \quad A^T \bar{\lambda} < \bar{\tau} c, \quad A \bar{x} = b, \quad (\bar{\tau}, \bar{x}) > 0.
\]

Hence, then point \((x, \lambda, s) = (\bar{x}/\bar{\tau}, \bar{\lambda}/\bar{\tau}, c - A^T \bar{\lambda}/\bar{\tau})\) is strictly feasible primal-dual point for (6). Therefore we obtain:

\[
0 < x^T s = c^T x - b^T \lambda = \frac{1}{\bar{\tau}} (c^T \bar{x} - b^T \bar{\lambda}) < 0,
\]

- Therefore SHSD can not be solved using feasible interior point method. However one may use infeasible interior point method.
**Simplified Hsd form**

**Theorem.** The primal-dual solution set $\Omega$ for (6) is nonempty iff all strictly complementary solutions $(\tau^*, x^*, \lambda^*, \kappa^*, s^*)$ of mLCP have $\tau^* > 0$ and $\kappa^* = 0$.

**Lemma.** Let $(\tau^*, x^*, \lambda^*, \kappa^*, s^*)$ be a strictly complementary solution of mLCP with $\tau^* > 0$ and $\kappa^* = 0$. Then $(x, \lambda, s) = \frac{1}{\tau^*}(x^*, \lambda^*, s^*)$ is a primal-dual solution of (6).

**Theorem.** Suppose that mLCP has a strictly complementary solution $(\tau^*, x^*, \lambda^*, \kappa^*, s^*)$ for which $\kappa^* > 0$. then at least one of $c^T x^*$ and $-b^T \lambda^*$ is negative and:

- if $c^T x^* < 0$, the dual problem is infeasible.
- If $-b^T \lambda^* < 0$, the primal problem is infeasible.
Given initial points for the primal-dual vector \((x^0, \lambda^0, s^0)\) with \(x^0 > 0\) and \(s^0 > 0\), the HSD for is define as:

$$\min \quad ((x^0)^T s^0 + 1) \theta$$

subject to

\[-c^T x + b^T \lambda \quad + \bar{z} \theta \quad \geq \quad 0,\]
\[c \tau - A^T \lambda \quad - \bar{c} \theta \quad \geq \quad 0,\]
\[-b \tau + Ax \quad + \bar{b} \theta \quad = \quad 0,\]
\[-\bar{z} \tau + \bar{c}^T x - \bar{b}^T \lambda \quad = \quad -((x^0)^T s^0 + 1),\]

\((\lambda, \theta)\) free, \((\tau, x) \geq 0,\)

where

\[\bar{b} = b - Ax^0, \quad \bar{c} = c - A^T \lambda^0 - s^0, \quad \bar{z} = c^T x^0 + 1 - b^T \lambda^0.\]
**HSD FORM**

- **HSD** problem is self-dual.
- For the equivalent mLCP formulation for HSD, we introduce slack variables \( \kappa \in \mathbb{R} \) and \( s \in \mathbb{R}^n \) for the first and second constraints, respectively. So we obtain:

\[
\begin{bmatrix}
\kappa \\
\mathbf{s} \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
0 & -c^T & b^T & \bar{z} \\
c & 0 & -A^T & -\bar{c} \\
-b & A & 0 & \bar{b} \\
-\bar{z} & \bar{c}^T & -\bar{b}^T & 0
\end{bmatrix}
\begin{bmatrix}
\tau \\
\mathbf{x} \\
\lambda \\
\theta
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
(x^0)^T s^0 + 1
\end{bmatrix}, \tag{7}
\]

\((\tau, \mathbf{x}, \kappa, s) \geq 0, \quad \tau \kappa + \mathbf{x}^T s = 0.\)

- \((\tau, \mathbf{x}, \lambda, \theta, \kappa, s) = (1, \mathbf{x}^0, \lambda^0, 1, 1, s^0).\) is a strictly feasible solution for above mLCP problem and \((\tau, \mathbf{x}, \lambda, \theta) = (1, \mathbf{x}^0, \lambda^0, 1)\) is a strictly feasible solution for HSD.