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Accelerated generalized SOR method for a class of complex systems of linear equations*

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Abstract. For solving a broad class of complex symmetric linear systems, recently Salkuyeh et al. recast the system in a real formulation and studied a generalized successive over-relaxation (GSOR) iterative method. In this paper, we introduce an accelerated GSOR (AGSOR) iterative method which involves two iteration parameters. Then, we theoretically study its convergence properties and determine its optimal iteration parameters and corresponding optimal convergence factor. Finally, some numerical computations are presented to validate the theoretical results and compare the performance of the AGSOR method with those of the GSOR and MHSS methods.

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Key words: Complex linear systems, Symmetric positive definite, Optimal parameters, Generalized SOR method, Accelerated GSOR method, MHSS.

1. Introduction

Consider the complex system of linear equations of the form

$$(W + iT)(x + iy) = b_1 + ib_2, \quad x, y, b_1, b_2 \in \mathbb{R}^n, \tag{1}$$

where $i = \sqrt{-1}$ and $W, T \in \mathbb{R}^{n\times n}$ are symmetric matrices with at least one of them being positive definite. In this paper, without loss of generality, we assume that W is symmetric positive definite. We can see such systems in many problems such as diffuse optical tomography [1], FFT-based solution of certain time-dependent PDEs [9], quantum mechanics [10], molecular scattering [13], structural dynamics [11], and lattice quantum chromodynamics [12]. The reader can refer to [8] for more examples and additional references.

Bai, Benzi and Chen in [3] considered the disadvantages arisen from the use of Hermitian and skew-Hermitian splitting (HSS) method [5] straightforwardly, and presented a modified version of the HSS iteration method (MHSS) as following:

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The MHSS iteration method: Given an initial guess $u^{(0)} \in \mathbb{C}^n$ and positive constant α, for $k = 0, 1, 2, \ldots$, until $\{u^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + W)u^{(k+\frac{1}{2})} = (\alpha I - iT)u^{(k)} + b, \\ (\alpha I + T)u^{(k+1)} = (\alpha I + iW)u^{(k+\frac{1}{2})} - ib, \end{cases} \tag{2}$$

where $u^{(k)} = x^{(k)} + iy^{(k)}$, $b = b_1 + ib_2$ and I is the identity matrix.

They have also showed that for any positive constant α the MHSS method converges unconditionally to the unique solution of the system of linear equations and considered MHSS-preconditioner to accelerate Krylov subspace methods such as GMRES or its restarted variant GMRES(l)[14]. The two half-steps at each MHSS iterate require solutions of two systems whose coefficient matrices are $\alpha I + W$ and $\alpha I + T$. Since both of these matrices are symmetric positive definite, one can exactly solve the corresponding systems by the Cholesky factorization and in the inexact version, by the conjugate gradient (CG) method.

In recent years, much work has been done in solving linear systems with 2×2 block structure, especially in the context of saddle point problems. For example, see the Uzawa method [7], the generalized SOR (GSOR) method [6], the accelerated HSS (AHSS) method [4]. It is possible to avoid complex arithmetic by rewriting the complex system (1) in the following real-valued form

$$\begin{bmatrix} W & -T \\ T & W \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{3}$$

Recently Salkuyeh, Hezari and Edalatpour in [15], by splitting the coefficient matrix of the system (3) into

$$\begin{bmatrix} W & -T \\ T & W \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -T & 0 \end{bmatrix} - \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}, \tag{4}$$

applied the generalized SOR iterative method to the equivalent real system (3) and introduced the following iterative method

$$\begin{cases} Wx^{(k+1)} = (1-\alpha)Wx^{(k)} + \alpha Ty^{(k)} + \alpha b_1, \\ Wy^{(k+1)} = -\alpha Tx^{(k+1)} + (1-\alpha)Wy^{(k)} + \alpha b_2, \end{cases} \tag{5}$$

where $0 \neq \alpha \in \mathbb{R}$. The two half-steps at each GSOR iterate require solutions with the matrix W that can be exactly solved by the Cholesky factorization or inexactly by the CG method. This is different from MHSS method, in which two linear subsystems with different coefficient matrices $\alpha I + W$ and $\alpha I + T$ need to be solved at every iteration step. Besides its use as a solver, the GSOR iteration has also been used as a preconditioner to accelerate Krylov subspace methods such as GMRES(l).

In the matrix-vector form, the GSOR iteration method can be equivalently rewritten as

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix} = \mathcal{G}_\alpha \begin{bmatrix} x^k \\ y^k \end{bmatrix} + \mathcal{C}_\alpha \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{6}$$

with

$$\mathcal{G}_\alpha = \begin{bmatrix} I & 0 \\ \alpha S & I \end{bmatrix}^{-1} \begin{bmatrix} (1-\alpha)I & \alpha S \\ 0 & (1-\alpha)I \end{bmatrix}, \quad \mathcal{C}_\alpha = \alpha \begin{bmatrix} W & 0 \\ \alpha T & W \end{bmatrix}^{-1},$$

where $S = W^{-1}T$. Here, $\mathcal{G}_\alpha$ is the iteration matrix of the GSOR method.

In [15], the authors presented convergence analysis and optimal iteration parameter and corresponding optimal convergence factor of the method as following

$$\alpha^* = \frac{2}{1+\sqrt{1+\gamma_{\max}^2}}, \quad \text{and} \quad \rho(G_{\alpha^*}) = 1-\alpha^* = \frac{\sqrt{1+\gamma_{\max}^2}-1}{\sqrt{1+\gamma_{\max}^2}+1}, \tag{7}$$

where $\gamma_{\max}$ are the largest eigenvalue of the matrix S in modulus.

In this paper, we propose a new version of the GSOR method to solve (3) in which two parameters α and β are incorporated. Hereafter, the new method is called accelerated GSOR method which is equivalent to the GSOR method, when $\alpha = \beta$. Hence, it is expected that the AGSOR method will converge faster than the GSOR method for suitable parameters. The two half-steps at each AGSOR iterate, like GSOR method, require solutions with the matrix W that can be exactly solved by the Cholesky factorization or inexactly by the CG method. We discuss the convergence analysis of the AGSOR method and obtain the optimal value of iteration parameters and corresponding optimal convergence factor.

The rest of the paper is organized as follows. In Section 2 we propose our method and investigate its convergence properties. Section 3 is devoted to some numerical experiments to show the effectiveness of the AGSOR iteration method. Finally, in Section 4, some concluding remarks are given.

2. Main Results

Let the parameters α and β be two nonzero real numbers and

$$\Upsilon = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix},$$

where I is the $n \times n$ identity matrix. Premultiplying both sides of Eq. (3) by Υ gives

$$\begin{bmatrix} \alpha W & -\alpha T \\ \beta T & \beta W \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha b_1 \\ \beta b_2 \end{bmatrix}. \tag{8}$$

For the coefficient matrix in (8), we make the splitting

$$\begin{bmatrix} \alpha W & -\alpha T \\ \beta T & \beta W \end{bmatrix} = D - E - F,$$

wherein

$$D = \begin{bmatrix} \alpha W & 0 \\ 0 & \beta W \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 \\ -\beta T & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & \alpha T \\ 0 & 0 \end{bmatrix}.$$

Then, similar to the GSOR method, the AGSOR iterative scheme is constructed as follows

$$\begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = G_{\alpha,\beta} \begin{bmatrix} x^{(k)} \\ y^{(k)} \end{bmatrix} + c, \tag{9}$$

where

$$G_{\alpha,\beta} = \begin{bmatrix} W & 0 \\ \beta T & W \end{bmatrix}^{-1} \begin{bmatrix} (1-\alpha)W & \alpha T \\ 0 & (1-\beta)W \end{bmatrix} = \begin{bmatrix} I & 0 \\ \beta S & I \end{bmatrix}^{-1} \begin{bmatrix} (1-\alpha)I & \alpha S \\ 0 & (1-\beta)I \end{bmatrix}, \tag{10}$$

is the iteration matrix of the AGSOR method and

$$c = \begin{bmatrix} W & 0 \\ \beta T & W \end{bmatrix}^{-1} \begin{bmatrix} \alpha b_1 \\ \beta b_2 \end{bmatrix},$$

with $S = W^{-1}T$. It is easy to verify that Eq. (9) is equivalent to

$$\begin{cases} Wx^{(k+1)} = (1-\alpha)Wx^{(k)} + \alpha T y^{(k)} + \alpha b_1, \\ Wy^{(k+1)} = -\beta T x^{(k+1)} + (1-\beta)Wy^{(k)} + \beta b_2. \end{cases} \tag{11}$$

At each iteration of the AGSOR method, two sub-systems with coefficient matrix W should be solved which can be done by the Cholesky factorization or inexactly by the CG algorithm. Obviously, when $\alpha = \beta$ the AGSOR iteration method reduces to the GSOR iteration method. Next, we discuss the convergence properties of the AGSOR method for solving the system (3).

Lemma 1. *[15] Let $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric, respectively. Then the eigenvalues of matrix $S = W^{-1}T$ are all real.*

The next lemma provides a functional relation between the eigenvalues of the matrix $S = W^{-1}T$ and the iteration matrix $G_{\alpha,\beta}$ of the AGSOR method.

Lemma 2. *Let α and β be two real numbers and $G_{\alpha,\beta}$ be the iteration matrix of the AGSOR method. Then, the eigenvalues of $G_{\alpha,\beta}$ are determined by the functional equation*

$$(1-\alpha-\lambda)(1-\beta-\lambda) = -\lambda\alpha\beta\gamma^2, \tag{12}$$

where γ and λ are the eigenvalues of $S = W^{-1}T$ and $G_{\alpha,\beta}$, respectively.

Proof. Let $\alpha \neq \beta$ and let λ be an eigenvalue of $G_{\alpha,\beta}$. Then, we must have

$$\begin{aligned} 0 = \det(G_{\alpha,\beta} - \lambda I) &= \det\left(\begin{bmatrix} (1-\alpha)I & \alpha S \\ 0 & (1-\beta)I \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \\ \beta S & I \end{bmatrix} \right) \\ &= \det\left(\begin{bmatrix} (1-\alpha-\lambda)I & \alpha S \\ -\lambda\beta S & (1-\beta-\lambda)I \end{bmatrix} \right). \end{aligned} \tag{13}$$

Now, if $\lambda \neq 1-\alpha$, then

$$\begin{aligned} \det(G_{\alpha,\beta} - \lambda I) &= \det((1-\alpha-\lambda)I)\det((1-\beta-\lambda)I + \frac{\lambda}{1-\alpha-\lambda}\alpha\beta S^2) \\ &= (1-\alpha-\lambda)^n \det((1-\beta-\lambda)I + \frac{\lambda}{1-\alpha-\lambda}\alpha\beta S^2). \end{aligned}$$

Therefore, in this case we have

$$\det((1-\beta-\lambda)I+\frac{\lambda}{1-\alpha-\lambda}\alpha\beta S^2)=0,$$

and the desired result is obtained. If $\lambda=1-\alpha$ then it follows from $\alpha\neq\beta$ that $\lambda\neq 1-\beta$, and similar to the previous case we can deduce the desired result. When $\alpha=\beta$, as mentioned, the AGSOR method reduses to the GSOR method. Hence according to [15, Theorem 1] we see that the eigenvalues of $G_{\alpha,\alpha}$ are determined by the functional equation

$$(1-\alpha-\lambda)^2=-\lambda\alpha^2\gamma^2, \tag{14}$$

which is the same as the equation (12), when $\alpha=\beta$. This completes the proof. □

Lemma 3. *[2, 16] Both roots of the real quadratic equation $\lambda^2-r\lambda+s=0$ are less than one in modulus if and only if $|s|<1$ and $|r|<1+s$.*

Concerning the convergence of the stationary AGSOR iteration method, we have the following theorem.

Theorem 1. *Let $W,T\in\mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric, respectively. Then the AGSOR method (9) is convergent for all b and c satisfying*

$$0<c<b<c\frac{1-\gamma_{\max}^2}{2}+2, \tag{15}$$

where $b=\alpha+\beta$, $c=\alpha\beta$ and $\gamma_{\max}$ is the largest eigenvalue of $S=W^{-1}T$ in modulus.

Proof. Suppose that λ is an arbitrary eigenvalue of $G_{\alpha,\beta}$. Using Lemma 2, there is an eigenvalue γ of S which satisfies Eq. (12). Since $b=\alpha+\beta$ and $c=\alpha\beta$, then α and β are the real roots of the quadratic equation $x^2-bx+c=0$, and Eq. (12) can be rewritten as

$$\lambda^2+(c\gamma^2+b-2)\lambda+c-b+1=0. \tag{16}$$

Now from Lemma 3, $|\lambda|<1$ if and only if

$$|c-b+1|<1, \tag{17}$$

and

$$|c\gamma^2+b-2|<c-b+2. \tag{18}$$

From (17) and (18), it is easy to obtain $0<c<b<c\frac{1-\gamma^2}{2}+2$. But, by noticing that $c\frac{1-\gamma^2}{2}+2$ is monotonically decreasing with respect to $|\gamma|$, and from Lemma 1, we must have

$$0<c<b<c\frac{1-\gamma_{\max}^2}{2}+2,$$

which completes the proof. □

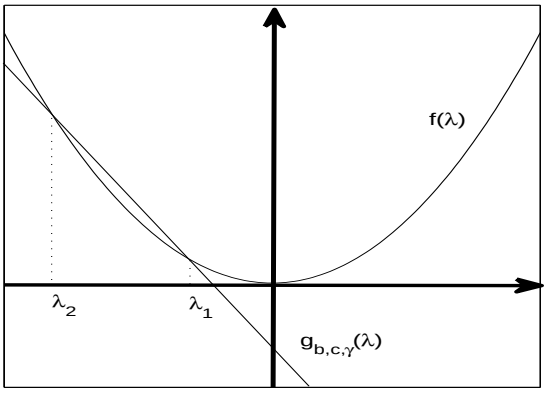

Figure 1: Graph of $f(\lambda)$ and $g_{b,c,\gamma}(\lambda)$.

In general, it is difficult to find the optimal values of α and β satisfying Eq. (15). Instead, we determine these parameters in a subset of the convergence region (15), such that besides their optimality in this subset, the corresponding convergence factor of the AGSOR method is smaller than that of the GSOR method. Let γ be an eigenvalue of $S = W^{-1}T$. In the proof of the Theorem 1, as seen Eq. (12) is equivalent to Eq. (16) which can be rewritten as

$$\lambda^2 = (2 - b - c\gamma^2)\lambda + b - c - 1,$$

where $b = \alpha + \beta$, $c = \alpha\beta$ and λ is an eigenvalue of iteration matrix $G_{\alpha,\beta}$. Set $f(\lambda) = \lambda^2$ and $g_{b,c,\gamma}(\lambda) = (2 - b - c\gamma^2)\lambda + b - c - 1$. So, the intersections of $f(\lambda)$ and $g_{b,c,\gamma}(\lambda)$ are the roots of Eq. (16) and we have

$$\lambda_{1,2}(b, c, \gamma) = \frac{(2 - b - c\gamma^2) \pm \sqrt{b^2 - 4c + c\gamma^2(c\gamma^2 + 2b - 4)}}{2}. \tag{19}$$

Now, suppose that $\gamma_{\min}$ and $\gamma_{\max}$ are the smallest and largest eigenvalues of the matrix S in modulus, respectively. Moreover, let $\bar{b}$ and $\bar{c}$ be in the convergence region (15) such that $g_{\bar{b},\bar{c},\gamma}(\lambda)$ is tangent to $f(\lambda)$ for $\gamma = \gamma_{\max}$. Then, from Eq. (19), we observe that

$$\bar{b}^2 - 4\bar{c} + \bar{c}\gamma_{\max}^2(\bar{c}\gamma_{\max}^2 + 2\bar{b} - 4) = 0. \tag{20}$$

Furthermore, suppose that $\bar{b}$ and $\bar{c}$ satisfy

$$(\gamma_{\max}^2 + \gamma_{\min}^2)\bar{c} + 2\bar{b} - 4 \geq 0. \tag{21}$$

From Eq. (20) we have

$$\bar{b} = -\bar{c}\gamma_{\max}^2 + 2\sqrt{\bar{c}(1 + \gamma_{\max}^2)}. \tag{22}$$

Note that $\bar{b} \neq -\bar{c}\gamma_{\max}^2 - 2\sqrt{\bar{c}(1 + \gamma_{\max}^2)}$, since $\bar{b}$ and $\bar{c}$ satisfy Eq. (15). Having in mind that the eigenvalues of A are all real, from Eq. (21), we see that

$$(\gamma_{\max}^2 + \gamma^2)\bar{c} + 2\bar{b} - 4 \geq 0 \quad \forall\gamma \in \sigma(S),$$

where $\sigma(S)$ denotes the spectrum of the matrix S. But, multiplying both sides of the latter inequality by $(\gamma_{\max}^2 - \gamma^2)$ results in

$$(\gamma_{\max}^4 - \gamma^4)\bar{c} + 2\bar{b}(\gamma_{\max}^2 - \gamma^2) - 4(\gamma_{\max}^2 - \gamma^2) \geq 0 \quad \forall \gamma \in \sigma(S),$$

which is equivalent to

$$\bar{b}^2 - 4\bar{c} + \bar{c}\gamma^2(\bar{c}\gamma^2 + 2\bar{b} - 4) \leq \bar{b}^2 - 4\bar{c} + \bar{c}\gamma_{\max}^2(\bar{c}\gamma_{\max}^2 + 2\bar{b} - 4) = 0 \quad \forall \gamma \in \sigma(S). \quad (23)$$

So, by (19) for any $\gamma \in \sigma(S)$, the eigenvalues $\lambda_{1,2}$ corresponding to γ, $\bar{b}$ and $\bar{c}$ are obtained as

$$\lambda_{1,2}(\bar{b}, \bar{c}, \gamma) = \frac{(2 - \bar{b} - \bar{c}\gamma^2) \pm i\sqrt{4\bar{c} - \bar{b}^2 - \bar{c}\gamma^2(\bar{c}\gamma^2 + 2\bar{b} - 4)}}{2}, \quad (24)$$

wherein $4\bar{c} - \bar{b}^2 - \bar{c}\gamma^2(\bar{c}\gamma^2 + 2\bar{b} - 4) \geq 0$, and the modulus of the eigenvalues $\lambda_{1,2}(\bar{b}, \bar{c}, \gamma)$ are given by

$$|\lambda_{1,2}(\bar{b}, \bar{c}, \gamma)| = \sqrt{1 + \bar{c} - \bar{b}} \quad \forall \gamma \in \sigma(S), \quad (25)$$

which is independent of the eigenvalues of S. Then, by substituting (22) into (25) we obtain

$$|\lambda_{1,2}(\bar{b}, \bar{c}, \gamma)| = |1 - \sqrt{\bar{c}(1 + \gamma_{\max}^2)}| \quad \forall \gamma \in \sigma(S). \quad (26)$$

Next, by combining the relations (15), (20) and (21), we define a subset $\Omega_{b,c}$ of the convergence region of the AGSOR method as follows

$$\Omega_{b,c} = \left\{ b, c \in \mathbb{R} \;\middle|\; \begin{array}{c} 0 < c < b < c\frac{1-\gamma_{\max}^2}{2} + 2, \\ b = -c\gamma_{\max}^2 + 2\sqrt{c(1+\gamma_{\max}^2)}, \\ (\gamma_{\max}^2 + \gamma_{\min}^2)c + 2b - 4 \geq 0 \end{array} \right\}.$$

Now, since $b = \alpha + \beta$ and $c = \alpha\beta$, we define the subset $\Psi_{\alpha,\beta}$ as

$$\Psi_{\alpha,\beta} = \Omega_{b,c}. \quad (27)$$

From the above discussion we conclude that if the parameters $\bar{\alpha}, \bar{\beta} \in \Psi_{\alpha,\beta}$, or equivalently $\bar{b}, \bar{c} \in \Omega_{b,c}$, then the spectral radius $\rho(G_{\bar{\alpha},\bar{\beta}})$ of the iteration matrix $G_{\bar{\alpha},\bar{\beta}}$ of the AGSOR method is equal to $|1 - \sqrt{\bar{c}(1 + \gamma_{\max}^2)}|$.

In the next theorem we introduce the optimal iteration parameters of the AGSOR method in $\Psi_{\alpha,\beta}$. Then, we determine the corresponding convergence factor of the iterative method.

Theorem 2. *Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric, respectively. Also, suppose that $G_{\alpha,\beta}$ is defined by Eq. (10). Set $b = \alpha + \beta$ and $c = \alpha\beta$. Moreover, let $\gamma_{\min}$ and $\gamma_{\max}$ be the smallest and largest eigenvalues of the matrix $S = W^{-1}T$ in modulus, respectively. Then, the optimal iteration parameters α^* and β^* of the AGSOR method in $\Psi_{\alpha,\beta}$, defined by (27), are given by*

$$\alpha^* = \frac{b^* + \sqrt{b^{*2} - 4c^*}}{2}, \quad \beta^* = \frac{b^* - \sqrt{b^{*2} - 4c^*}}{2}, \quad (28)$$

wherein

$$b^* = 4\frac{1+\sqrt{(1+\gamma_{\min}^2)(1+\gamma_{\max}^2)}}{(\sqrt{\gamma_{\max}^2+1}+\sqrt{\gamma_{\min}^2+1})^2}, \quad c^* = 4\frac{1}{(\sqrt{\gamma_{\max}^2+1}+\sqrt{\gamma_{\min}^2+1})^2}. \quad (29)$$

Also, the corresponding optimal convergence factor of the AGSOR method is given by

$$\rho(G_{\alpha^*,\beta^*}) = \frac{\sqrt{\gamma_{\max}^2+1}-\sqrt{\gamma_{\min}^2+1}}{\sqrt{\gamma_{\max}^2+1}+\sqrt{\gamma_{\min}^2+1}}, \quad (30)$$

where $\rho(G_{\alpha,\beta})$ is the spectral radius of the iteration matrix.

Proof. Consider $\bar{b}, \bar{c} \in \Omega_{b,c}$ and let us define $H_{b,c} = G_{\alpha,\beta}$. Therefore, $H_{b^*,c^*} = G_{\alpha^*,\beta^*}$ and obtaining the optimal parameters α^* and β^* of the AGSOR method is equivalent to obtaining b^* and c^* satisfying

$$(b^*, c^*) = \underset{(\bar{b},\bar{c})\in\Omega_{b,c}}{\operatorname{argmin}} \ |1-\sqrt{\bar{c}(1+\gamma_{\max}^2)}|. \quad (31)$$

Combining (20) and (21) results in

$$(\gamma_{\max}^2-\gamma_{\min}^2)^2\bar{c}^2 - 8(\gamma_{\max}^2+\gamma_{\min}^2+2)\bar{c}+16 \le 0,$$

and therefore, we must have

$$4\frac{1}{(\sqrt{\gamma_{\max}^2+1}+\sqrt{\gamma_{\min}^2+1})^2} \le \bar{c} \le 4\frac{1}{(\sqrt{\gamma_{\max}^2+1}-\sqrt{\gamma_{\min}^2+1})^2}, \quad (32)$$

and the problem (31) is converted to the following one

$$(b^*, c^*) \quad = \underset{4\frac{1}{(\sqrt{\gamma_{\max}^2+1}+\sqrt{\gamma_{\min}^2+1})^2} \le c \le 4\frac{1}{(\sqrt{\gamma_{\max}^2+1}-\sqrt{\gamma_{\min}^2+1})^2}}{\operatorname{arg\,min}} |1-\sqrt{c(1+\gamma_{\max}^2)}|.$$

To solve this problem, according to Fig. 2, we conclude that

$$c^* = 4\frac{1}{(\sqrt{\gamma_{\max}^2+1}+\sqrt{\gamma_{\min}^2+1})^2}, \quad (33)$$

also, by (22) and (33) we obtain

$$b^* = 4\frac{1+\sqrt{(1+\gamma_{\min}^2)(1+\gamma_{\max}^2)}}{(\sqrt{\gamma_{\max}^2+1}+\sqrt{\gamma_{\min}^2+1})^2}. \quad (34)$$

It is easy to verify that the values of b^* and c^* are in $\Omega_{b,c}$. Also, α^* and β^* are determined by the equations $b^* = \alpha^* + \beta^*$ and $c^* = \alpha^*\beta^*$ as given in (28). Finally, from Eq. (24) the optimal convergence factor of the AGSOR method in $\Psi_{\alpha,\beta}$ is obtained as

$$\rho(G_{\alpha^*,\beta^*}) = \frac{\sqrt{\gamma_{\max}^2+1}-\sqrt{\gamma_{\min}^2+1}}{\sqrt{\gamma_{\max}^2+1}+\sqrt{\gamma_{\min}^2+1}}, \quad (35)$$

which completes the proof. □

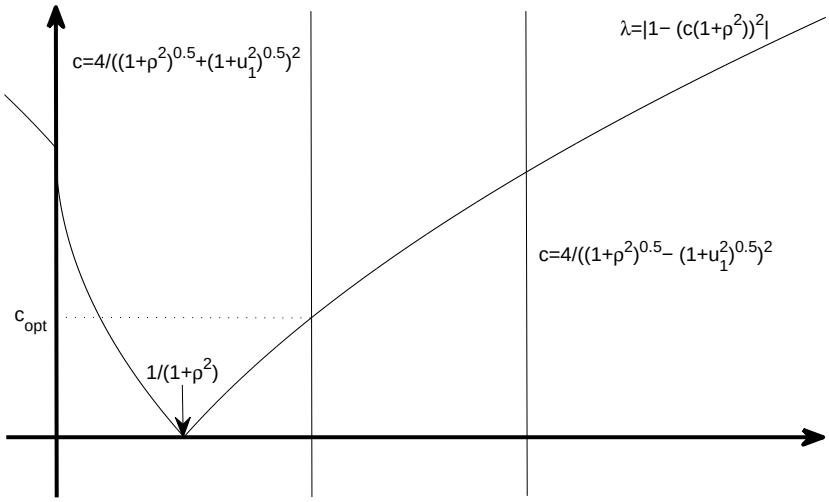

Figure 2: Graph of the function $\lambda = |1 - \sqrt{c(1+\gamma_{\max}^2)}|$ and the optimal value of c.

An immediate implication of this theorem is that the spectral radius of the iteration matrix corresponding to the optimal parameters depends to the extreme eigenvalues of the matrix $S = W^{-1}T$, for both the GSOR and AGSOR methods. In addition, in the case that the smallest eigenvalue of S is nonzero, from Eqs. (7) and (30) it follows that the corresponding optimal convergence factor of the AGSOR iteration method is smaller than that of the GSOR method.

Remark 1. *Under assumptions of Theorem 2, if the matrix $S = W^{-1}T$ is singular, then $\gamma_{\min} = 0$ and from Eqs. (29) and (28) it follows that $\alpha^* = \beta^* = \frac{2}{1+\sqrt{1+\gamma_{\max}^2}}$ and the AGSOR method reduces to the GSOR method.*

3. Numerical experiments

In this section, we use three test problems from [3] and an example of [9], to illustrate the feasibility and effectiveness of the AGSOR iteration method when is employed as a solver to solve the equivalent real system (3). We also compare the performance of the AGSOR method with those of the GSOR and MHSS methods, in terms of both the number of iterations (denoted as IT) and the total computing times (in seconds, denoted by CPU). In each iteration of both the AGSOR and GSOR iteration methods, we use the Cholesky factorization of the coefficient matrices to solve the sub-systems. The reported CPU times are the sum of the CPU times for the convergence of the method and the CPU times for computing the Cholesky factorization.

All the numerical experiments presented in this section were computed in double precision and the algorithms were implemented in MATLAB 7.12.0 and tested on a

64-bit 1.73 GHz intel Q740 core i7 processor and 4GB RAM running Windows 7. We use a null vector as an initial guess and the stopping criterion

$$\frac{\|b - Au^{(k)}\|_2}{\|b\|_2} < 10^{-6},$$

where $u^{(k)} = x^{(k)} + iy^{(k)}$.

Example 1. (see [3]) Consider the system of linear equations

$$\left[\left(K + \frac{3-\sqrt{3}}{\tau} I\right) + i\left(K + \frac{3+\sqrt{3}}{\tau} I\right)\right] x = b, \tag{36}$$

where τ is the time step-size and K is the five-point centered difference matrix approximating the negative Laplacian operator $L = -\Delta$ with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0,1] \times [0,1]$ with the mesh-size $h = \frac{1}{m+1}$. The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$, with $V_m = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. Hence, K is an $n \times n$ block-tridiagonal matrix, with $n = m^2$. We take

$$W = K + \frac{3-\sqrt{3}}{\tau} I \quad \text{and} \quad T = K + \frac{3-\sqrt{3}}{\tau} I,$$

and the right-hand side vector b with its jth entry b_j being given by

$$b_j = \frac{(1-i)j}{\tau(j+1)^2}, \quad j = 1, 2, \ldots, n.$$

In our tests, we take $\tau = h$. Furthermore, we normalize the coefficient matrix and the right-hand side vector by multiplying both by h^2.

Example 2. (see [3]) Consider the system of linear equations $(W + iT)x = b$, with

$$T = I \otimes V + V \otimes I \quad \text{and} \quad W = 10(I \otimes V_c + V_c \otimes I) + 9(e_1 e_m^T + e_m e_1^T) \otimes I,$$

where $V = \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$, $V_c = V - e_1 e_m^T - e_m e_1^T \in \mathbb{R}^{m\times m}$ and e_1 and e_m are the first and last unit vectors in $\mathbb{R}^m$, respectively. We take the right-hand side vector b to be $b = (1+i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. Here T and W correspond to the five-point centered difference matrices approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions and periodic boundary conditions, respectively, on a uniform mesh in the unit square $[0,1] \times [0,1]$ with the mesh-size $h = \frac{1}{m+1}$. Although this problem is an artificially constructed one, it is quite challenging for iterative solvers and therefore we include it in our tests.

Example 3. (see [3]) Consider the system of linear equations

$$\left[(-\omega^2 M + K) + i(\omega C_V + C_H)\right] x = b,$$

where M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, respectively, and ω is the driving circular

frequency. We take $C_H = \mu K$ with μ a damping coefficient, $M = I$, $C_V = 10I$, and K the five-point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0,1] \times [0,1]$ with the mesh-size $h = \frac{1}{m+1}$. The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$, with $V_m = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. Hence, K is an $n \times n$ block-tridiagonal matrix, with $n = m^2$. In this example for $\mu = .5$, $\mu = 1$ and $\mu = 2$, we set $\omega = \pi$ and the right-hand side vector b to be $b = (1 + i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. As before, we normalize the system by multiplying both sides through by h^2.

Example 4. (see [9]) Helmholtz equations are of fundamental importance in the modeling of wave propagation phenomena. In this example we consider the finite-difference discretization of the partial differential equation

$$-\nabla.(c\nabla u) + \sigma_1 u + i\sigma_2 u = f, \tag{37}$$

where the coefficients c, σ_1 and σ_2 are real-valued functions, u satisfies Dirichlet boundary conditions and $i = \sqrt{-1}$. We consider Eq. (37) on the 2D domain $[0,1] \times [0,1]$ with different values of c, $\sigma_1 = 1000$ and $\sigma_2 = 10000$. We discretize the problem with finite differences on a $m \times m$ grid with mesh size $h = 1/(m + 1)$. This leads to a system of linear equations

$$((cK + \sigma_1 I) + i\sigma_2 I)\, x = b,$$

where $K = I \otimes V_m + V_m \otimes I$ is a standard second-order finite-difference discretization of the diffusion operator $-\triangle u$, wherein $V_m = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. The right-hand side vector b is taken to be $b = (1 + i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. Furthermore, before solving the system we normalize the coefficient matrix and the right-hand side vector by multiplying both by h^2.

In Table 1, we report the optimal parameters of the MHSS, the GSOR and the AGSOR methods for Examples 1-4, for each choice of the spatial mesh-sizes. The optimal values of the parameter α (denoted by α_{exp}) for the MHSS iteration method are experimentally resulted in the least numbers of iterations for the method for each of the numerical examples and for different values of m. The optimal values of the parameters α and β (denoted by α^* and β^*) for the GSOR and the AGSOR methods have been obtained from the relations (7) and (28) in which the largest and smallest eigenvalues of matrix S can be estimated by a few iterations of the power method. The optimal convergence factor corresponding to the optimal parameters of the methods are listed in Table 2. Also, the iteration number and the CPU times of the MHSS, the GSOR and the AGSOR iteration methods for the examples are shown in Tables 3-6.

In Example 2, the smallest eigenvalue of $S = W^{-1}T$ is small for all values of m (about 0.1) and as expected, the optimal convergence factors of the GSOR and AGSOR methods are almost the same (see Table 2) and as a result it is also true in terms of both the number of iterations and the CPU times (see Table 4). In Examples 1 and 3, the results show that in general the AGSOR method is superior to the other methods in terms of the optimal convergence factor, iteration count

Table 1: The optimal parameters of the GSOR and AGSOR iteration methods for Examples 1-4.

Example	Method	Grid				
		16×16	32×32	64×64	128×128	256×256
No. 1	MHSS (α_{exp})	1.06	0.75	0.54	0.40	0.30
	GSOR (α^*)	0.550	0.495	0.457	0.432	0.428
	AGSOR (α^*,β^*)	$(0.880, 0.276)$	$(0.828, 0.244)$	$(0.791, 0.221)$	$(0.764, 0.209)$	$(0.749, 0.203)$
No. 2	MHSS (α_{exp})	0.21	0.08	0.04	0.02	0.01
	GSOR (α^*)	0.908	0.776	0.566	0.353	0.199
	AGSOR (α^*,β^*)	$(0.924, 0.892)$	$(0.798, 0.754)$	$(0.588, 0.545)$	$(0.369, 0.339)$	$(0.209, 0.189)$
No. 3, $\mu = 0.5$	MHSS (α_{exp})	0.56	0.31	0.16	0.08	0.04
	GSOR (α^*)	0.376	0.377	0.377	0.377	0.377
	AGSOR (α^*,β^*)	$(0.543, 0.248)$	$(0.541, 0.251)$	$(0.539, 0.252)$	$(0.539, 0.253)$	$(0.539, 0.253)$
$\mu = 1$	MHSS (α_{exp})	0.77	0.41	0.21	0.11	0.06
	GSOR (α^*)	0.317	0.318	0.318	0.319	0.319
	AGSOR (α^*,β^*)	$(0.614, 0.144)$	$(0.611, 0.146)$	$(0.611, 0.146)$	$(0.610, 0.146)$	$(0.610, 0.146)$
$\mu = 2$	MHSS (α_{exp})	0.98	0.53	0.28	0.15	0.08
	GSOR (α^*)	0.241	0.242	0.242	0.242	0.242
	AGSOR (α^*,β^*)	$(0.707, 0.062)$	$(0.703, 0.063)$	$(0.702, 0.063)$	$(0.702, 0.063)$	$(0.702, 0.063)$
No. 4, $c = 1$	MHSS (α_{exp})	6.33	2.21	0.93	0.42	0.13
	GSOR (α^*)	0.184	0.184	0.184	0.184	0.184
	AGSOR (α^*,β^*)	$(0.737, 0.031)$	$(0.430, 0.072)$	$(0.246, 0.137)$	$(0.199, 0.171)$	$(0.190, 0.178)$
$c = 0.1$	MHSS (α_{exp})	4.01	1.19	0.71	0.50	0.10
	GSOR (α^*)	0.181	0.181	0.181	0.181	0.181
	AGSOR (α^*,β^*)	$(0.990, 0.012)$	$(0.911, 0.018)$	$(0.624, 0.041)$	$(0.341, 0.092)$	$(0.219, 0.150)$
$c = 0.01$	MHSS (α_{exp})	4.01	1.19	0.71	0.50	0.10
	GSOR (α^*)	0.181	0.181	0.181	0.181	0.181
	AGSOR (α^*,β^*)	$(1, 0.010)$	$(0.998, 0.011)$	$(0.980, 0.013)$	$(0.845, 0.023)$	$(0.507, 0.055)$

and CPU times (see Tables 2, 3 and 5). In Fig. 3 we have compared the smallest eigenvalue of $S = W^{-1}T$ for different values of m and c for Example 4. As seen, the AGSOR method outperforms GSOR, especially when the value of c decreases (see Table 6). The cause of such performance is predictable by considering the relation (30) and Fig. 3. However, this performance becomes weaker as m increases.

In the first three examples, the AGSOR method behaves much better than the MHSS method in terms of both the number of iterations and CPU times. In Example 4, as seen from 6 for some values of m the number of iterations of the AGSOR method is greater than that of the MHSS method. Nevertheless, the cost per iteration of the AGSOR method is lower and leads to faster convergence in terms of CPU time.

4. Conclusion

By considering a new splitting of the coefficient matrix, we have presented a variant of the generalized SOR iterative method called accelerated GSOR method to solve the equivalent real formulation of complex linear system (3), where W is symmetric positive definite and T is symmetric. Convergence properties of the method has

Table 2: The optimal convergence factor corresponding to the optimal parameters of GSOR and AGSOR iteration methods presented in Table 1.

Example	Method	Grid				
		16×16	32×32	64×64	128×128	256×256
No. 1	GSOR	0.450	0.505	0.543	0.568	0.572
	AGSOR	0.294	0.360	0.404	0.432	0.448
No. 2	GSOR	0.092	0.224	0.434	0.647	0.801
	AGSOR	0.091	0.223	0.433	0.646	0.800
No. 3, $\mu = 0.5$	GSOR	0.624	0.623	0.623	0.623	0.623
	AGSOR	0.586	0.587	0.587	0.587	0.587
$\mu = 1$	GSOR	0.683	0.682	0.682	0.681	0.681
	AGSOR	0.575	0.576	0.577	0.577	0.577
$\mu = 2$	GSOR	0.759	0.758	0.758	0.758	0.758
	AGSOR	0.526	0.528	0.528	0.528	0.528
No. 4, $c = 1$	GSOR	0.816	0.816	0.816	0.816	0.816
	AGSOR	0.504	0.730	0.807	0.815	0.816
$c = 0.1$	GSOR	0.819	0.819	0.819	0.819	0.819
	AGSOR	0.099	0.296	0.600	0.774	0.815
$c = 0.01$	GSOR	0.819	0.819	0.819	0.819	0.819
	AGSOR	0.011	0.041	0.142	0.389	0.682

also been investigated. Some numerical examples have been presented to show the effectiveness of the method. Our numerical examples show that our method is significantly superior to GSOR method when the smallest eigenvalue of $S = W^{-1}T$ is not very small and the relation (30) confirms this.

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References

[1] S.R. ARRIDGE, *Optical tomography in medical imaging*, Inverse Probl. **15**(1999)41-93.
[2] O. AXELSSON, *Iterative Solution Methods*, Cambridge University Press, Cambridge, 1996.
[3] Z.-Z. BAI, M. Benzi, F. Chen, *Modified HSS iteration methods for a class of complex symmetric linear systems*, Computing **87**(2010)93-111.

Table 3: Numerical results of Example 1.

Method	$m \times m$	16×16	32×32	64×64	128×128	256×256
MHSS	IT	40	54	73	98	133
	CPU	0.034	0.114	0.612	3.468	21.273
GSOR	IT	19	22	24	26	27
	CPU	0.003	0.010	0.053	0.255	1.153
AGSOR	IT	15	17	19	21	22
	CPU	0.002	0.007	0.041	0.210	0.934

Table 4: Numerical results of Example 2.

Method	$m \times m$	16×16	32×32	64×64	128×128	256×256
MHSS	IT	44	76	130	246	468
	CPU	0.040	0.242	1.191	8.788	118.432
GSOR	IT	7	11	20	35	71
	CPU	0.002	0.011	0.064	0.474	3.970
AGSOR	IT	7	11	20	34	70
	CPU	0.002	0.011	0.063	0.470	3.971

[4] Z.-Z. Bai and G.H. Golub, *Accelerated Hermitian and skew-Hermitian splitting iteration methods for saddle-point problems*, IMA J. Numer. Anal. **27**(2007)1-23.

[5] Z.-Z. Bai, G.H. Golub, M.K. Ng, *Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems*, SIAM. J. Matrix Anal. Appl. **24**(2003)603-626.

[6] Z.-Z. Bai, B.N. Parlett and Z.-Q. Wang, *On generalized successive overrelaxation methods for augmented linear systems*, Numer. Math. **102**(2005)1-38.

[7] Z.-Z. Bai, Z.-Q. Wang, *On parameterized inexact Uzawa methods for generalized saddle point problems, Linear Algebra Appl.* **428** (2008) 2900-2932.

[8] M. Benzi, D. Bertaccini, *Block preconditioning of real-valued iterative algorithms for complex linear systems*, IMA J. Numer. Anal. **28**(2008)598-618.

[9] D. Bertaccini, *Efficient solvers for sequences of complex symmetric linear systems*, Electr. Trans. Numer. Anal. **18**(2004)49-64.

[10] W.V. Dijk, F.M. Toyama, *Accurate numerical solutions of the time-dependent Schrdinger equation*, Phys. Rev. E **75**, 036707 (2007).

[11] A. Feriani, F. Perotti, V. Simoncini, *Iterative system solvers for the frequency analysis of linear mechanical systems*, Comput. Methods Appl. Mech. Eng. **190**(2000)1719-1739.

[12] A. Frommer, T. Lippert, B. Medeke, K. Schilling (eds), *Numerical challenges in lattice quantum chromodynamics*, Lecture notes in computational science and engineering **15**(2000)66-83.

[13] B. Poirier, *Efficient preconditioning scheme for block partitioned matrices with structured sparsity*, Numer. Linear Algebra Appl. **7**(2000)715-726.

Table 5: Numerical results of Example 3

μ	Method		Grid				
			16×16	32×32	64×64	128×128	256×256
$\mu = 0.5$	MHSS	IT	51	85	150	279	529
		CPU	0.039	0.171	1.190	9.600	95.093
	GSOR	IT	37	35	33	31	30
		CPU	0.007	0.015	0.066	0.321	1.19
	AGSOR	IT	33	31	30	30	30
		CPU	0.005	0.013	0.064	0.308	1.20
$\mu = 1$	MHSS	IT	56	94	168	306	555
		CPU	0.041	0.184	1.339	10.265	99.887
	GSOR	IT	46	43	41	38	37
		CPU	0.008	0.027	0.086	0.386	1.569
	AGSOR	IT	32	31	31	31	31
		CPU	0.006	0.018	0.066	0.304	1.293
$\mu = 2$	MHSS	IT	56	95	169	303	552
		CPU	0.041	0.199	1.371	10.323	87.970
	GSOR	IT	64	60	57	53	51
		CPU	0.012	0.034	0.112	0.494	2.120
	AGSOR	IT	28	27	28	28	28
		CPU	0.005	0.015	0.060	0.259	1.290

[14] Y. Saad, M.H. Schultz, GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems, SIAM J. Sci. Stat. Comput. **7**(1986)856-869.

[15] D. K. Salkuyeh, D. Hezari, V. Edalatpour, Generalized SOR iterative method for a class of complex symmetric linear system of equations, International Journal of Computer Mathematics, 2014, DOI: 10.1080/00207160.2014.912753.

[16] D.M. Young, *Iterative Solutions of Large Linear Systems*, Academic Press, New York, 1971.

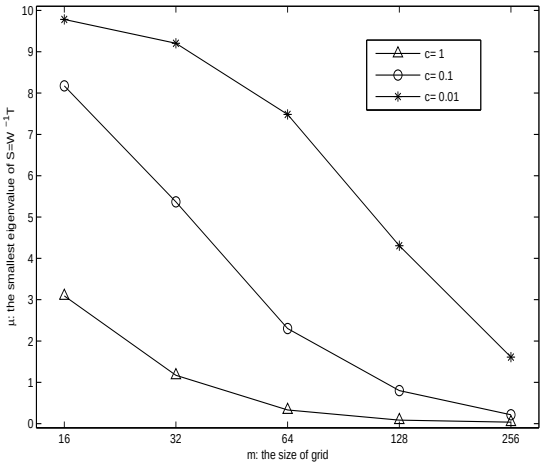

Figure 3: The smallest eigenvalue of $S = W^{-1}T$ for different values of m and c for Example 4.

Table 6: Numerical results of Example 4 when $\sigma_1 = 1000$ and $\sigma_2 = 10000$

c	Method		Grid				
			16×16	32×32	64×64	128×128	256×256
$c = 1$	MHSS	IT	30	30	30	30	34
		CPU	0.016	0.051	0.218	0.844	4.476
	GSOR	IT	97	94	93	92	91
		CPU	0.013	0.039	0.179	0.914	4.298
	AGSOR	IT	29	62	88	92	91
		CPU	0.005	0.028	0.169	0.916	4.278
$c = 0.1$	MHSS	IT	31	31	31	31	31
		CPU	0.017	0.052	0.222	0.926	4.066
	GSOR	IT	86	86	86	86	86
		CPU	0.012	0.037	0.164	0.872	4.105
	AGSOR	IT	9	16	35	67	76
		CPU	0.002	0.008	0.073	0.689	3.681
$c = 0.01$	MHSS	IT	31	31	31	31	31
		CPU	0.017	0.052	0.220	0.952	4.410
	GSOR	IT	87	87	87	86	86
		CPU	0.012	0.037	0.180	0.880	4.144
	AGSOR	IT	5	7	11	21	49
		CPU	0.002	0.005	0.028	0.262	2.462