A generalization of the Gauss-Seidel iteration method for solving absolute value equations

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Abstract. Based on the Gauss-Seidel splitting, we present a new matrix splitting iteration method, called generalized Gauss-Seidel (GGS) iteration method, for solving the large sparse absolute value equation (AVE) $A\mathbf{x} - |\mathbf{x}| = \mathbf{b}$ where $A \in \mathbb{R}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}^n$ and investigate its convergence properties. Moreover, by preconditioning AVE, a preconditioned variant of the GGS (PGGS) method is presented. Numerical experiments illustrate the efficiency of both GGS and PGGS iterations.

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1 Introduction

Consider the absolute value equation (AVE)

$$A\mathbf{x} - |\mathbf{x}| = \mathbf{b}, \tag{1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $\mathbf{b} \in \mathbb{R}^n$ and $|\cdot|$ denotes the absolute value. Some frequently occurring optimization problems such as linear programming, convex quadratic programming and linear complementarity problem (LCP) [2, 3, 4], can be equivalently reformulated as AVE (1) [11, 12, 13]. For example, for a given $M \in \mathbb{R}^{n\times n}$ and $\mathbf{q} \in \mathbb{R}^n$, the LCP consists of finding a vector $\mathbf{z} \in \mathbb{R}^n$ such that

$$\mathbf{z} \geq 0, \quad M\mathbf{z} + \mathbf{q} \geq 0, \quad \mathbf{z}^T(M\mathbf{z} + \mathbf{q}) = 0, \tag{2}$$

and when the matrix $M - I$ is nonsingular, the LCP (2) can be reduced to the following AVE (see [11])

$$(M - I)^{-1}(M + I)\mathbf{x} - |\mathbf{x}| = (M - I)^{-1}\mathbf{q},$$

with

$$\mathbf{x} = \frac{1}{2}((M - I)\mathbf{z} + \mathbf{q}).$$

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Mangasarian and Meyer in [11] established existence and nonexistence of a solution for the absolute value equations and showed that if all singular values of A exceed 1 or $\|A^{-1}\|_2 < 1$, then for every right-hand side vector $\mathbf{b}$, the AVE (1) has a unique solution where $\|.\|_2$ is the 2-norm. Also, in [14] an algorithm is given for computing all of the solutions of AVE (1).

Generally, the problem (1) is NP-hard, since the general NP-hard linear complementarity problem (LCP) can be formulated as an AVE (1) (see [3]). In [10] a direct generalized Newton method was proposed for solving the AVE (1) when the singular values of A exceed 1, that guaranteed existence and uniqueness of a solution for the AVE (1). In each iteration of this method the following linear system is solved

$$\left(A - D(\mathbf{x}^{(k)})\right)\mathbf{x}^{(k+1)} = \mathbf{b},$$

where $\mathbf{x}^{(0)}$ is an initial guess and $D(\mathbf{x}^{(k)})$ is a diagonal matrix of the form $D(\mathbf{x}^{(k)}) = \text{diag}(\text{sign}(\mathbf{x}^{(k)}))$ wherein $\text{sign}(\mathbf{x})$ denotes a vector with components equal to 1, 0 or -1 depending on whether the corresponding component of $\mathbf{x}$ is positive, zero or negative, respectively. Rohn et al. in [15] proposed another method to solve AVE (1) where in practice it is reduced to the well known Picard iteration method which can be summarized as

$$\mathbf{x}^{(k+1)} = A^{-1}\left(|\mathbf{x}^{(k)}| + \mathbf{b}\right), \qquad k = 0, 1, 2, \ldots, \tag{3}$$

where $\mathbf{x}^{(0)} = A^{-1}\mathbf{b}$ is the initial guess. In addition, they presented a sufficient condition as $\rho(|A^{-1}|) < 1$ for the uniquely solvability of the equation (1) and compared this condition with the Mangasarian-Meyer condition $\sigma_{\max}(A^{-1}) < 1$ given in [11], where $\rho(|A^{-1}|)$ and $\sigma_{\max}(A^{-1})$ show the spectral radius of $|A^{-1}|$ and the greatest singular value of A^{-1}, respectively. When the AVE (1) is infeasible, the optimum correction of this equation has been investigated by Ketabchi et al. in [7, 8]. Actually, the authors made changes in both the coefficient matrix and the right-hand side vector of AVE (1) and obtained the following minimization problem

$$\begin{aligned} &\min_{\mathbf{x}} \min_{E,\mathbf{r}} (\|E\|^2 + \|\mathbf{r}\|^2) \\ &s.t. \quad (A+E)\mathbf{x} - |\mathbf{x}| = \mathbf{b} + \mathbf{r}, \end{aligned}$$

where $E \in \mathbb{R}^{n\times n}$ is a perturbation matrix and $\mathbf{r} \in \mathbb{R}^n$ is a perturbation vector. Then, they obtained the optimum pair $(E^*, \mathbf{r}^*)$ of the above problem by solving a fractional objective function of two quadratic functions as follows

$$\min_{\mathbf{x}\in\mathbb{R}^n} \frac{\|A\mathbf{x} - |\mathbf{x}| - \mathbf{b}\|^2}{1 + \|\mathbf{x}\|^2}. \tag{4}$$

Also, they used a feasible direction method [7] and a genetic algorithm [8] to solve the problem (4).

Recently, Salkuyeh [17] proposed the Picard-HSS iteration method to solve AVE (1) when A is a non-symmetric positive definite matrix as following:

The Picard-HSS iteration method*: Let the matrix $A \in \mathbb{C}^{n\times n}$ be positive definite with $H = \frac{1}{2}(A + A^H)$ and $S = \frac{1}{2}(A - A^H)$ being the Hermitian and skew-Hermitian parts of A, respectively. Given an initial guess $\boldsymbol{x}^{(0)} \in \mathbb{C}^n$ and a sequence $\{l_k\}_{k=0}^{\infty}$ of positive integers, compute $\boldsymbol{x}^{(k+1)}$ for $k = 0, 1, 2, \ldots$, using the following iteration scheme until $\{\boldsymbol{x}^{(k)}\}$ satisfies the stopping criterion:*

(a) Set $\boldsymbol{x}^{(k,0)} := \boldsymbol{x}^{(k)}$;

(b) For $\ell = 0, 1, \ldots, l_k - 1$, solve the following linear systems to obtain $\boldsymbol{x}^{(k,\ell+1)}$:

$$\begin{cases} (\alpha I + H)\boldsymbol{x}^{(k,\ell+\frac{1}{2})} = (\alpha I - S)\boldsymbol{x}^{(k,\ell)} + |\boldsymbol{x}^{(k)}| + \boldsymbol{b}, \\ (\alpha I + S)\boldsymbol{x}^{(k,\ell+1)} = (\alpha I - H)\boldsymbol{x}^{(k,\ell+\frac{1}{2})} + |\boldsymbol{x}^{(k)}| + \boldsymbol{b}, \end{cases}$$

where α is a given positive constant.

(c) Set $\boldsymbol{x}^{(k+1)} := \boldsymbol{x}^{(k,l_k)}$.

Moreover, he obtained a sufficient condition for the convergence of the above algorithm and presented some numerical examples that show inability of the Picard and generalized Newton methods to converge for several cases while the Picard-HSS method properly converges to the exact solution.

In this paper, based on the the Gauss-Seidel splitting, we present a new matrix splitting iteration method, called generalized Gauss-Seidel (GGS) iteration method to solve the AVE (1). Let

$$A = D - E - F, \tag{5}$$

where D, E and F are diagonal, strictly lower triangular and strictly upper triangular matrices, respectively. We assume that the diagonal entries of A are nonzero. In each iteration of the GGS method for solving AVE (1), a lower triangular absolute value system

$$(D - E)\mathbf{x} - |\mathbf{x}| = \mathbf{c}, \tag{6}$$

should be solved where $\mathbf{c}$ is a fixed vector. Under the condition that the diagonal entries of A exceed unit, we will present an algorithm to solve system (6). Moreover, we consider the Gauss-Seidel iteration method to solve preconditioned AVE (1) and suggest an efficient preconditioner to expedite the convergence rate of the method when the coefficient matrix A is a Z-matrix. For this reason, we transform the AVE (1) into the preconditioned form

$$P_\beta A\mathbf{x} - P_\beta|\mathbf{x}| = P_\beta\mathbf{b}, \tag{7}$$

where $P_\beta = D + \beta F$. The GGS iteration method corresponding to the preconditioned system will be referred to as the preconditioned GGS (PGGS) iteration method. It is noteworthy that the motivation of choosing this preconditioner stems from [9] in which Kotakemori et al. applied the preconditioner P_β for solving the linear system $A\mathbf{x} = \mathbf{b}$ with the Gauss-Seidel method when A is a strictly diagonally dominant Z-matrix.

The rest of the paper is organized as follows. We present some necessary notations and useful lemmas in Section 2 and establish the generalized Gauss-Seidel iteration method to solve AVE (1) and its convergence properties in Section 3. The Gauss-Seidel iteration method to solve preconditioned AVE (7) is considered in Section 4 and some numerical experiments are given in Section 5. The paper is ended by giving some concluding remarks in Section 6.

2 Preliminaries

We briefly introduce some necessary notations, definitions and lemmas. For a matrix $A \in \mathbb{R}^{m\times n}$, the infinity norm of A is defined as $\|A\|_\infty = \max_{1\le i\le m}\sum_{j=1}^{n}|a_{ij}|$. The matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is said to be strictly row diagonally dominant if $\sum_{j=1,j\neq i}^{n}|a_{ij}| < |a_{ii}|$, $i = 1, 2, \ldots, n$. For two

matrices $A = (a_{ij})$ and $B = (b_{ij}) \in \mathbb{R}^{n\times n}$, we write $A \geq 0$ ($A > 0$) if $a_{ij} \geq 0$ ($a_{ij} > 0$) holds for all $1 \leq i, j \leq n$ and we write $A \geq B$ ($A > B$) if $A - B \geq 0$ ($A - B > 0$). We use $|A| = (|a_{ij}|)$ to denote the absolute value of the matrix A.

The square matrix $A = (a_{ij})$ is called a Z-matrix if its off-diagonal entries are all nonpositive. The matrix A is called an M-matrix if it is a Z-matrix and nonsingular with $A^{-1} \geq 0$. A is said to be an H-matrix if its comparison matrix $\langle A\rangle = (c_{ij})$ is an M-matrix, where

$$c_{ij} = \begin{cases} |a_{ij}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

Moreover, a matrix A is called an H_+-matrix if it is an H-matrix having positive diagonal entries.

For later use we recall some lemmas and theorems.

Lemma 1. *[18] Let A be an M-matrix of order n and $A \leq B$. If $b_{ij} \leq 0$, $i \neq j$, then $B = (b_{ij})$ is also an M-matrix.*

Lemma 2. *[1] Let M be an H-matrix and let $\langle M\rangle$ be the comparison matrix. Then $|M^{-1}| \leq \langle M\rangle^{-1}$.*

Definition 1. For $n \times n$ real matrices A, M and N, $A = M - N$ is a regular splitting of the matrix A if M is nonsingular with $M^{-1} > 0$ and $N > 0$.

Lemma 3. *[9] Let $A = M - N$ is a regular splitting of the matrix A. Then $\rho(M^{-1}N) < 1$ if and only if A is nonsingular and $A^{-1} \geq 0$.*

Theorem 1. *[18] Let G be a nonnegative matrix. Then $\rho(G) < 1$ if and only if $I-G$ is nonsingular and $(I - G)^{-1}$ is nonnegative.*

Theorem 2. *[20] For every $B \in \mathbb{C}^{n\times n}$ and all $\epsilon > 0$, there exists a norm $\|.\|$ on $\mathbb{C}^n$ such that for the corresponding induced norm, $\|B\| \leq \rho(B) + \epsilon$.*

Lemma 4. *[19, Page 46] Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be points in $\mathbb{R}^n$. Then $\||\boldsymbol{x}| - |\boldsymbol{y}|\|_\infty \leq \|\boldsymbol{x} - \boldsymbol{y}\|_\infty$.*

3 The Generalized Gauss-Seidel method to solve AVE

By rewriting the AVE (1) as the fixed-point equation

$$(D - E)\mathbf{x} - |\mathbf{x}| = F\mathbf{x} + \mathbf{b}, \tag{8}$$

we can define the generalized Gauss-Seidel (GGS) iteration for solving (1) as follows

$$(D - E)\mathbf{x}^{(k+1)} - |\mathbf{x}^{(k+1)}| = F\mathbf{x}^{(k)} + \mathbf{b}, \quad k = 0, 1, 2, \ldots, \tag{9}$$

where $\mathbf{x}^{(0)}$ is a given initial guess.

Let $D(\mathbf{x}) = \text{diag}(\text{sign}(\mathbf{x}))$. Without loss of generality we can assume that the diagonal entries of A are positive (otherwise we set $\mathbf{x} = D(\text{diag}(A))\mathbf{y}$ and consider the equation $AD(\text{diag}(A))\mathbf{y} - |\mathbf{y}| = \mathbf{b}$). We first give the following lemma.

Lemma 5. *Let $a, b \in \mathbb{R}$ and $a > 1$. Equation $ax - |x| = b$ has only one solution in $\mathbb{R}$. If $b \geq 0$, then the solution is given by $x = \frac{b}{a-1}$, otherwise $x = \frac{b}{a+1}$ is the solution.*

Proof. The proof is easy and is omitted here. □

In the sequel, based on Lemma 5, we present Algorithm 1 for solving each step of the GGS iteration method (9) when diagonal entries of the matrix A are greater than one.

Algorithm 1. Generalized Gauss-Seidel Iteration Method

1. For $k = 0, 1, \ldots,$ until convergence, Do
2. Set $s = b_1$
3. For $i = 1, 2, \ldots, n,$ Do
4. If $s \geq 0$, then
5. $x_i^{(k+1)} := s/(a_{ii} - 1)$
6. Else
7. $x_i^{(k+1)} := s/(a_{ii} + 1)$
8. EndIf
9. set $s = b_{i+1} - \sum_{j=1}^{i} a_{ij} x_j^{(k+1)} + \sum_{j=i+2}^{n} a_{ji} x_j^{(k)}$
10. EndDo
11. EndDo

Now, we are ready to describe the convergence properties of the proposed method.

Theorem 3. *Suppose that the AVE (1) is solvable. Let the diagonal entries of A are greater than one and the matrix $D - E - I$ is strictly row diagonally dominant. If*

$$\|(D-E)^{-1}F\|_\infty < 1 - \|(D-E)^{-1}\|_\infty, \tag{10}$$

then the equation (1) has a unique solution $\boldsymbol{x}^*$ and the sequence $\{\boldsymbol{x}^{(k)}\}$ obtained from (9) converges to the solution $\boldsymbol{x}^*$.

Proof. We first show that $\|(D-E)^{-1}\|_\infty < 1$. Obviously, if $E = 0$, then $\|(D-E)^{-1}\|_\infty = \|D^{-1}\|_\infty < 1$ since the diagonal entries of D greater than 1. Hence, we assume that $E \neq 0$. Since the diagonal entries of A are greater than 1 and $D - E - I$ is strictly row diagonally dominant, we obtain

$$0 \leq |E|e < (D - I)e,$$

or, equivalently

$$D^{-1}e < (I - |L|)e, \tag{11}$$

where, $L = D^{-1}E$ and $e = (1, \ldots, 1)^T$. Also, we have

$$\begin{aligned} 0 \leq |(I-L)^{-1}| &= |I + L + L^2 + \cdots + L^{n-1}| \\ &\leq (I + |L| + |L|^2 + \cdots + |L|^{n-1}) = (I - |L|)^{-1}. \end{aligned} \tag{12}$$

Therefore, from Eqs. (11) and (12) we obtain

$$\begin{aligned}|(D-E)^{-1}|e &= |(I-L)^{-1}D^{-1}|e \leq |(I-L)^{-1}||D^{-1}|e \\ &< (I-|L|)^{-1}(I-|L|)e = e.\end{aligned}$$

This implies

$$\|(D-E)^{-1}\|_\infty < 1. \tag{13}$$

To prove uniqueness of the solution, let $\mathbf{x}^*$ and $\mathbf{y}^*$ be two different solutions of the AVE (1). Then, from Eq. (8) we see that

$$\begin{aligned}\mathbf{x}^* &= (D-E)^{-1}|\mathbf{x}^*| + (D-E)^{-1}(F\mathbf{x}^* - \mathbf{b}), \\ \mathbf{y}^* &= (D-E)^{-1}|\mathbf{y}^*| + (D-E)^{-1}(F\mathbf{y}^* - \mathbf{b}).\end{aligned} \tag{14}$$

But, from Eq. (10), Lemma 4 and the latter equations, we obtain

$$\begin{aligned}\|\mathbf{x}^* - \mathbf{y}^*\|_\infty &\leq \|(D-E)^{-1}\|_\infty \||\mathbf{x}^*| - |\mathbf{y}^*|\|_\infty + \|(D-E)^{-1}F\|_\infty \|\mathbf{x}^* - \mathbf{y}^*\|_\infty \\ &< \|(D-E)^{-1}\|_\infty \|\mathbf{x}^* - \mathbf{y}^*\|_\infty + (1 - \|(D-E)^{-1}\|_\infty)\|\mathbf{x}^* - \mathbf{y}^*\|_\infty \\ &= \|\mathbf{x}^* - \mathbf{y}^*\|_\infty,\end{aligned}$$

which is contradiction. Therefore, $\mathbf{x}^* = \mathbf{y}^*$.

To prove the convergence of the method let $\mathbf{x}^*$ be the unique solution of the AVE (1). Therefore, from (14) and

$$\mathbf{x}^{(k+1)} = (D-E)^{-1}|\mathbf{x}^{(k+1)}| + (D-E)^{-1}(F\mathbf{x}^{(k)} - \mathbf{b}), \tag{15}$$

we deduce

$$\mathbf{x}^{(k+1)} - \mathbf{x}^* = (D-E)^{-1}(|\mathbf{x}^{(k+1)}| - |\mathbf{x}^*|) + (D-E)^{-1}F(\mathbf{x}^{(k)} - \mathbf{x}^*).$$

Taking infinity norm in the latter equation yields

$$\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\|_\infty - \|(D-E)^{-1}\|_\infty \||\mathbf{x}^{(k+1)}| - |\mathbf{x}^*|\|_\infty \leq \|(D-E)^{-1}F\|_\infty \|\mathbf{x}^{(k)} - \mathbf{x}^*\|_\infty.$$

Then, from Lemma 4 we get

$$\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\|_\infty - \|(D-E)^{-1}\|_\infty \|\mathbf{x}^{(k+1)} - \mathbf{x}^*\|_\infty \leq \|(D-E)^{-1}F\|_\infty \|\mathbf{x}^{(k)} - \mathbf{x}^*\|_\infty,$$

and since $|(D-E)^{-1}\|_\infty \leq 1$ it follows that

$$\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\|_\infty \leq \frac{\|(D-E)^{-1}F\|_\infty}{1 - \|(D-E)^{-1}\|_\infty} \|\mathbf{x}^{(k)} - \mathbf{x}^*\|.$$

This inequality shows that if the condition (10) is fulfilled then the convergence of the method is deduced. □

From Theorem 3 the next corollary can be stated.

Corollary 1. *Suppose that the AVE (1) is solvable. Let $A-I$ be an strictly row diagonally dominant matrix having positive diagonal entries. If*

$$\|(D-E)^{-1}F\|_\infty < 1 - \|(D-E)^{-1}\|_\infty, \tag{16}$$

then the equation (1) has a unique solution $\boldsymbol{x}^*$ and the sequence $\{\boldsymbol{x}^{(k)}\}$ obtained from (9) converges to the solution $\boldsymbol{x}^*$.

Now, we are going to establish the convergence of the method when the matrix A of AVE (1) is an H-matrix.

Theorem 4. *Let the AVE (1) be solvable and $A-I$ be an H_+-matrix. Then, the sequence $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ generated by the iteration method (9) converges to the unique solution $\boldsymbol{x}^*$ of the AVE (1) for any initial vector $\boldsymbol{x}^{(0)}$.*

Proof. We have

$$\langle A\rangle = \langle D-E\rangle - |F|,$$

so, we obtain

$$\langle A-I\rangle \leq \langle A\rangle \leq \langle D-E\rangle \leq \mathrm{diag}(D-E) = D,$$

hence, by Lemma 1, $D-E$ is an H-matrix and by Lemma 2 it holds that

$$|(D-E)^{-1}| \leq \langle D-E\rangle^{-1}. \tag{17}$$

Now, from (9) and since $\mathbf{x}^*$ is the solution of AVE (1), we get

$$\mathbf{x}^{(k+1)} - \mathbf{x}^* = (D-E)^{-1}(|\mathbf{x}^{(k+1)}| - |\mathbf{x}^*|) + (D-E)^{-1}F(\mathbf{x}^{(k)} - \mathbf{x}^*)$$

By taking absolute values on both sides of the above equation and using the inequality (17) it follows that

$$|\mathbf{x}^{(k+1)} - \mathbf{x}^*| \leq |(D-E)^{-1}(|\mathbf{x}^{(k+1)}| - |\mathbf{x}^*|)| + |(D-E)^{-1}F(\mathbf{x}^{(k)} - \mathbf{x}^*)| \tag{18}$$

$$\leq |\langle D-E\rangle^{-1}||\mathbf{x}^{(k+1)} - \mathbf{x}^*| + |\langle D-E\rangle^{-1}||F||\mathbf{x}^{(k)} - \mathbf{x}^*|. \tag{19}$$

Let $G = \langle D-E\rangle^{-1}$. Since $D-E$ is an H-matrix, it follows that G is an M-matrix and hence $G^{-1} \geq 0$. On the other hand, obviously $\rho(G) < 1$. Therefore, thanks to Theorem 1 we deduce that $I-G$ is nonsingular and $(I-G)^{-1} \geq 0$. Hence, from Eq. (18) we get

$$|\mathbf{x}^{(k+1)} - \mathbf{x}^*| \leq (I - \langle D-E\rangle^{-1})^{-1}\langle D-E\rangle^{-1}|F||\mathbf{x}^{(k)} - \mathbf{x}^*|,$$

or equivalently

$$|\mathbf{x}^{(k+1)} - \mathbf{x}^*| \leq (\langle D-E\rangle - I)^{-1}|F||\mathbf{x}^{(k)} - \mathbf{x}^*|.$$

Letting

$$\hat{M} = \langle D-E\rangle - I, \qquad \hat{N} = |F|,$$

evidently, the iteration sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to $\mathbf{x}^*$ if the spectral radius of the matrix $\hat{L} = \hat{M}^{-1}\hat{N}$ is less than one, where $\hat{L}$ is the iteration matrix corresponding to the matrix splitting $\hat{A} = \hat{M} - \hat{N}$. As the diagonal entries of A exceed unit we have $\hat{A} = \langle A\rangle - I = \langle A-I\rangle$ and $\hat{M} = \langle D-E-I\rangle$. So, $\hat{A}$ is an M-matrix and since $\hat{A} < \hat{M}$ then $\hat{M}$ is also an M-matrix by using the Lemma 1. Also, as $\hat{N} > 0$ it follows that the splitting $\hat{A} = \hat{M} - \hat{N}$ is a regular splitting. Thus, by the Theorem 2, $\rho(\hat{L}) < 1$ holds true. This completes the proof. □

4 Preconditioned generalized Gauss-Seidel

In this section, we express the PGGS iteration method for solving the AVE (1) while the matrix A is a Z-matrix. Consider the preconditioned AVE (7) and let

$$P_\beta A = \tilde{D} - \tilde{E} - \tilde{F}, \tag{20}$$

in which $\tilde{D}$ is a diagonal matrix, $\tilde{E}$ is a strictly lower triangular matrix and $\tilde{F}$ is a strictly upper triangular matrix. Hence, the PGGS iteration method for solving (1) is defined as follows

$$(\tilde{D} - \tilde{E})\mathbf{x}^{(k+1)} - D|\mathbf{x}^{(k+1)}| = -\beta F|\mathbf{x}^{(k)}| + \tilde{F}\mathbf{x}^{(k)} + P_\beta \mathbf{b}, \quad k = 0, 1, 2, \ldots, \tag{21}$$

or equivalently

$$D^{-1}(\tilde{D} - \tilde{E})\mathbf{x}^{(k+1)} - |\mathbf{x}^{(k+1)}| = -\beta D^{-1}F|\mathbf{x}^{(k)}| + D^{-1}\tilde{F}\mathbf{x}^{(k)} + D^{-1}P_\beta \mathbf{b}, \quad k = 0, 1, 2, \ldots, \tag{22}$$

where $\mathbf{x}^{(0)}$ is a given initial guess.

Let

$$FE = \bar{D} - \bar{E} - \bar{F}, \tag{23}$$

where $\bar{D}$, $-\bar{E}$ and $-\bar{F}$ are the diagonal, strictly lower triangular and strictly upper triangular matrices. Then, from $P_\beta = D + \beta F$ and Eq. (20), we have

$$\tilde{D} = (D^2 - \beta\bar{D}), \quad \tilde{E} = (DE - \beta\bar{E}) \quad \text{and} \quad \tilde{F} = (DF - \beta FD - \beta\bar{F} + \beta F^2). \tag{24}$$

Note that, when the diagonal entries of $D^{-1}(\tilde{D} - \tilde{E})$ exceed 1, $\mathbf{x}^{(k+1)}$ can be obtained by Algorithm 1 by a simple modification. In the next lemma, we show that, if A is a Z-matrix with diagonal entries greater than 1, then the diagonal entries of $D^{-1}(\tilde{D} - \tilde{E})$ exceed 1.

Lemma 6. *Let $A = (a_{ij})$ be a Z-matrix having diagonal entries greater than 1. Assume that*

$$0 < \beta < \frac{a(a-1)}{\max_i d_{ii}}, \tag{25}$$

where $a = \min_{1\le i\le n} a_{ii}$ and $\bar{D} = diag(d_{11}, d_{22}, \ldots, d_{nn})$. Then the diagonal entries of $D^{-1}(\tilde{D} - \tilde{E})$ are greater than 1.

Proof. It is enough to show that the diagonal entries of $H = D^{-1}\tilde{D}$ exceed 1. Suppose that $H = (h_{ij})$. From (24) we obtain

$$h_{ii} = \frac{a_{ii}^2 - \beta d_{ii}}{a_{ii}}. \tag{26}$$

Also, from (23) we deduce that

$$d_{ii} = \sum_{k=i+1}^{n} a_{ik}a_{ki}, \tag{27}$$

and since A is a Z-matrix, for each i, $d_{ii} \geq 0$. Now, if $\max_i d_{ii} = 0$, then it follows from (26) that for all i, $h_{ii} = a_{ii} > 1$, otherwise from (25) for all i we have

$$\beta < \frac{a(a-1)}{\max_i d_{ii}} \leq \frac{a_{ii}(a_{ii}-1)}{d_{ii}}, \tag{28}$$

and by Eq. (26) the i-th diagonal entry of H is obtained as

$$h_{ii} = \frac{a_{ii}^2 - \beta d_{ii}}{a_{ii}} > a_{ii} - \frac{a_{ii}(a_{ii}-1)}{d_{ii}} \frac{d_{ii}}{a_{ii}} > 1.$$

Therefore the diagonal entries of H are greater than 1. □

The next theorem presents sufficient conditions for the convergence of the PGGS iteration method.

Theorem 5. *Let the conditions of Lemma 6 hold and $P_\beta = D + \beta F$, where $\beta \in \mathbb{R}$. Let also $D - E - I$ be strictly row diagonally dominant. There exists an $r > 0$ such that if $0 < |\beta| < r$, then $\|(\tilde{D} - \tilde{E})^{-1} D\|_\infty < 1$. Also, if the AVE (1) is solvable and*

$$\beta \|(\tilde{D} - \tilde{E})^{-1} F\|_\infty + \|(\tilde{D} - \tilde{E})^{-1} \tilde{F}\|_\infty \leq 1 - \|(\tilde{D} - \tilde{E})^{-1} D\|_\infty, \tag{29}$$

then the solution of the equation AVE (1) is unique and the PGGS iteration method (22) converges to the solution.

Proof. Similar to Theorem 1, we first show that there exists an $r > 0$ such that $0 < |\beta| < r$ results in

$$\|(\tilde{D} - \tilde{E})^{-1} D\|_\infty < 1. \tag{30}$$

To this end from Eq. (24) we have

$$\begin{aligned}(\tilde{D} - \tilde{E})^{-1} D &= (D^2 - \beta \bar{D} - DE + \beta \bar{E})^{-1} D = (D(D-E) - \beta(\bar{D} - \bar{E}))^{-1} D \\ &= ((D-E) - \beta D^{-1}(\bar{D} - \bar{E}))^{-1} = (I - \beta(D-E)^{-1} D^{-1}(\bar{D} - \bar{E}))^{-1} (D-E)^{-1},\end{aligned}$$

which results in

$$\|(\tilde{D} - \tilde{E})^{-1} D\|_\infty \leq \|(I - \beta(D-E)^{-1} D^{-1}(\bar{D} - \bar{E}))^{-1}\|_\infty \|(D-E)^{-1}\|_\infty.$$

If $\beta = 0$, then from (13) we have $\|(\tilde{D} - \tilde{E})^{-1} D\|_\infty \leq \|(D-E)^{-1}\|_\infty < 1$. On the other hand, from continuity of $\|.\|_\infty$, there exists an $r > 0$ such that for $0 < |\beta| < r$ the relationship (30) holds true. In the sequel, we choose

$$0 < \beta < \min\{r, \frac{a(a-1)}{\max_i d_{ii}}\}.$$

From (22) for $k = 0, 1, 2, \ldots$, we have

$$\begin{aligned}\mathbf{x}^{(k+1)} =& (\tilde{D} - \tilde{E})^{-1} D |\mathbf{x}^{(k+1)}| + \beta (\tilde{D} - \tilde{E})^{-1} F |\mathbf{x}^{(k)}| \\ &+ (\tilde{D} - \tilde{E})^{-1} (\tilde{F} \mathbf{x}^{(k)} - P_\beta \mathbf{b}).\end{aligned} \tag{31}$$

If $\mathbf{x}^*$ is the solution of (1), then

$$\mathbf{x}^* = (\tilde{D} - \tilde{E})^{-1} D |\mathbf{x}^*| + \beta (\tilde{D} - \tilde{E})^{-1} F |\mathbf{x}^*| + (\tilde{D} - \tilde{E})^{-1} (\tilde{F} \mathbf{x}^* - P_\beta \mathbf{b}), \tag{32}$$

Subtracting Eq. (32) from Eq. (31) yields

$$\begin{aligned}\mathbf{x}^{(k+1)} - \mathbf{x}^* =&(\tilde{D} - \tilde{E})^{-1}D(|\mathbf{x}^{(k+1)}| - |\mathbf{x}^*|)\\ &+ \beta(\tilde{D} - \tilde{E})^{-1}F(|\mathbf{x}^{(k)}| - |\mathbf{x}^*|) + (\tilde{D} - \tilde{E})^{-1}\tilde{F}(\mathbf{x}^{(k)} - \mathbf{x}^*),\end{aligned}$$

and we obtain the following inequality

$$\begin{aligned}&\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\|_\infty - \|(\tilde{D} - \tilde{E})^{-1}D\|_\infty\||\mathbf{x}^{(k+1)}| - |\mathbf{x}^*|\|_\infty\\ &\quad\le \beta\|(\tilde{D} - \tilde{E})^{-1}F\|_\infty\||\mathbf{x}^{(k)}| - |\mathbf{x}^*|\|_\infty + \|(\tilde{D} - \tilde{E})^{-1}\tilde{F}\|_\infty\|\mathbf{x}^{(k)} - \mathbf{x}^*\|_\infty.\end{aligned}$$

Now, from Lemma 4, we get

$$(1 - \|(\tilde{D} - \tilde{E})^{-1}D\|_\infty)\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\|_\infty \le (\beta\|(\tilde{D} - \tilde{E})^{-1}F\|_\infty + \|(\tilde{D} - \tilde{E})^{-1}\tilde{F}\|_\infty)\|\mathbf{x}^{(k)} - \mathbf{x}^*\|_\infty,$$

or

$$\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\|_\infty \le \frac{\beta\|(\tilde{D} - \tilde{E})^{-1}F\|_\infty + \|(\tilde{D} - \tilde{E})^{-1}\tilde{F}\|_\infty}{1 - \|(\tilde{D} - \tilde{E})^{-1}D\|_\infty}\|\mathbf{x}^{(k)} - \mathbf{x}^*\|_\infty.$$

Therefore, if the relationship (29) is fulfilled, then the PGGS iteration method (22) is convergent. Proof of the uniqueness is the same as that of Theorem 3. □

It seems that it is not easy to state a theorem to compare the performance of the GGS and PGGS iterative methods to solve AVE from theoretical point of view. However, as will be seen in the section of the numerical experiments the PGGS method is more effective than the GGS method.

5 Numerical experiments

In this section we give some numerical experiments to show the effectiveness of the GGS and PGGS iteration methods to solve the AVE (1). All the numerical experiments presented in this section were computed in double precision and the algorithms were implemented in MATLAB 8.0.0.783 (64-bit) and tested on a 64-bit 1.73 GHz intel Q740 core i7 processor and 4GB RAM running Windows 7. We use a null vector as an initial guess and the stopping criterion

$$\frac{\|A\mathbf{x}^{(k)} - |\mathbf{x}^{(k)}| - \mathbf{b}\|_2}{\|\mathbf{b}\|_2} \le 10^{-7},$$

is always used where $\mathbf{x}^{(k)}$ is the computed solution by each of the methods at iterate k. For the Picard-HSS iteration method, the stopping criterion

$$\frac{\|\mathbf{b} - A\mathbf{s}^{(k,\ell)}\|_2}{\|\mathbf{b}\|_2} \le 0.01,$$

and a maximum number of iterations 10 ($l_k = 10$, $k = 0, 1, 2, \ldots$) for the inner iterations are used. In all the examples the right-hand side vector of AVE (1) is taken such a way that the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ with

$$x_i = (-1)^i i, \quad i = 1, 2, \ldots, n,$$

be the exact solution. In the implementation of the Picard-HSS and the PGGS iteration methods, the optimal parameters have been obtained experimentally, since the computation of the optimal parameter is often problem-dependent and generally difficult to be determined. In the Picard-HSS iteration method subsystems with the coefficient matrix $(\alpha I + H)$ are solved by the Cholesky factorization of the coefficient matrix and subsystems with the coefficient matrix $(\alpha I + S)$ are solved by the LU factorization of the coefficient matrix or the MATLAB backslash operator.

We give three examples and the corresponding numerical results are adjusted in four tables. In the presented tables we give the number of iterations for the convergence (denoted by IT) and CPU times for the convergence (denoted by CPU). Here, we mention that the reported CPU times are the sum of the CPU times for the convergence and the CPU times for computing the Cholesky and LU factorizations of the coefficient matrices and the CPU times for constructing the preconditioner for the PGGS method. It is mentioned that the CPU times are given in seconds.

Example 1. Let

$$M = \begin{pmatrix} S & -I & -I & 0 & \cdots & 0 & 0 \\ 0 & S & -I & -I & \cdots & 0 & 0 \\ 0 & 0 & S & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & -I \\ \vdots & \vdots & \ddots & \ddots & \ddots & S & -I \\ 0 & 0 & \ddots & \ddots & 0 & 0 & S \end{pmatrix} \in \mathbb{R}^{m^3 \times m^3}, \tag{33}$$

where $S = \text{tridiag}(1, 8, -1) \in \mathbb{R}^{m^2 \times m^2}$ and $I \in \mathbb{R}^{m^2 \times m^2}$ is the identity matrix. Note that $M - I_n$ is an H-matrix with diagonal entries greater than 1, where I_n is the identity matrix of size m^3. For AVE (1) when $A = M$, we present the numerical results for different values of m in Table 1. This table shows that in terms of the CPU time the GGS and the PGGS methods are the best one among the tested methods. This is the case for the generalized Newton method in terms of the iterations. We also observe that the iteration numbers for all the methods are independent of the size of the problem as m increases.

In the sequel, we consider the two-dimensional convection diffusion equation (see [17])

$$\begin{cases} -(u_{xx} + u_{yy}) + q(u_x + u_y) + pu &= f(x, y) \quad (x, y) \in \Omega, \\ u(x, y) &= 0 \qquad (x, y) \in \partial\Omega, \end{cases}$$

where $\Omega = (0, 1) \times (0, 1)$, $\partial\Omega$ its boundary, and q is a positive constant and p is a real number. We use the five-point finite difference scheme to the diffusive terms and the central difference scheme to the convective terms. Let $h = 1/(m + 1)$ and $Re = (qh)/2$ denote the equidistant step size and the mesh Reynolds number, respectively. Then, we get a system of linear equations $Bx = d$, where B is a matrix of order $n = m^2$ of the form

$$B = T_x \otimes I + I \otimes T_y + pI_n, \tag{34}$$

wherein I_m and I_n are, respectively, the identity matrices of order m and n, $\otimes$ means the Kronecker product symbol, and T_x and T_y are the tridiagonal matrices

$$T_x = \text{tridiag}(t_2, t_1, t_3) \quad \text{and} \quad T_y = \text{tridiag}(t_2, 0, t_3),$$

Table 1: Numerical results for Example 1 for different values of m.

Method	m	10	15	20	25	30
Picard-HSS	IT	9	9	9	9	9
	CPU	0.188	0.628	1.978	3.878	8.751
	α_{exp}	10	10	10	10	10
Gen. Newton	IT	3	3	3	3	3
	CPU	0.033	0.063	0.159	0.373	0.846
Picard	IT	9	9	9	9	9
	CPU	0.016	0.123	0.551	1.647	3.754
GGS	IT	13	13	13	13	13
	CPU	0.004	0.009	0.015	0.026	0.044
PGGS	IT	7	7	7	7	7
	CPU	0.002	0.008	0.013	0.023	0.039
	β_{exp}	1.1	1.1	1.1	1.1	1.1

with

$$t_1 = 4, \quad t_2 = -1 - Re, \quad t_3 = -1 + Re.$$

It is easy to see that the matrix B is in general non-symmetric positive definite. We define the matrix A in AVE (1) by making use of the matrix B for our numerical experiments as follows.

Example 2. Let $q = 0$ and $p = 0, -1$ and set

$$A = B + 0.5(L - L^T),$$

where the matrix B is provided by (34) and L is the strictly lower part of B. We present the numerical results for different values of n in Table 2.

The reported results in Table 2 show that for $p = 0$ the PGGS method is superior to the other methods in terms of the CPU time and in terms of the number of the iterations the Generalized Newton method is the best. For $p = -1$, as seen the PGGS method is the best one among the tested methods in terms of the CPU times and number of the iterations. Another observation which can be posed here is that the generalized Newton method and the Picard methods fail to converge in 10000 iterations (denoted by "Fail"). Finally, in this example we see that the propose preconditioner is very effective to expedite the speed of the convergence of the GGS method.

Example 3. In this example, we set $A = B$ where B is defined by (34). Numerical results for different values of n ($n = 400, 1600, 3600, 6400, 10000$), different values of q ($q = 1, 10, 100$) and p ($p = 0, -1$) are given in Tables 3 and 4. These tables show that PGGS method always converge, but four other methods fail to converge for several cases. For the rest of the cases, almost all of the previous observations can be posed here.

Table 2: Numerical results for Example 2 for different values of n and $q = 0$ and p.

p	Method	n	400	1600	3600	6400	10000
0	Picard-HSS	IT	61	117	174	236	296
		CPU	0.086	0.748	4.360	14.137	32.698
		α_{exp}	7.9	8.9	9.1	9.6	9.8
	Gen. Newton	IT	9	19	29	39	49
		CPU	0.027	0.139	0.605	1.506	3.354
	Picard	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	GGS	IT	84	166	253	342	433
		CPU	0.006	0.036	0.123	0.244	0.420
	PGGS	IT	15	27	53	69	86
		CPU	0.003	0.010	0.030	0.055	0.091
		β_{exp}	1.8	1.7	1.6	1.6	1.6
-1	Picard-HSS	IT	18	25	34	41	51
		CPU	0.020	0.124	0.817	1.904	4.291
		α_{exp}	9	9	8.9	8.7	8.7
	Gen. Newton	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	Picard	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	GGS	IT	67	96	123	149	174
		CPU	0.005	0.024	0.059	0.101	0.171
	PGGS	IT	10	13	16	19	21
		CPU	0.002	0.003	0.011	0.018	0.030
		β_{exp}	1.2	1.2	1.2	1.2	1.2

Table 3: Numerical results for Example 3 for different values of n and q and $p = 0$.

q	Method	n	400	1600	3600	6400	10000
1	Picard-HSS	IT	32	31	30	29	29
		CPU	0.033	0.168	0.602	1.645	4.171
		$\alpha_{\exp}$	6.8	6.8	6.8	6.8	6.8
	Gen. Newton	IT	5	5	5	5	5
		CPU	0.009	0.037	0.100	0.184	0.335
	Picard	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	GGS	IT	112	112	115	118	121
		CPU	0.013	0.016	0.056	0.083	0.122
	PGGS	IT	18	18	18	18	18
		CPU	0.003	0.004	0.012	0.016	0.024
		$\beta_{\exp}$	1.3	1.3	1.3	1.3	1.3
10	Picard-HSS	IT	71	33	38	37	33
		CPU	0.075	0.169	0.708	1.714	3.311
		$\alpha_{\exp}$	5.5	6.7	7.7	7.9	8
	Gen. Newton	IT	6	5	4	5	5
		CPU	0.020	0.039	0.082	0.181	0.345
	Picard	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	GGS	IT	296	128	135	127	122
		CPU	0.021	0.027	0.068	0.092	0.118
	PGGS	IT	25	20	22	20	19
		CPU	0.002	0.004	0.015	0.022	0.024
		$\beta_{\exp}$	2	1.4	1.4	1.4	1.4
100	Picard-HSS	IT	17	36	76	151	308
		CPU	0.019	0.178	1.399	6.278	23.299
		$\alpha_{\exp}$	5	5	5	5	5
	Gen. Newton	IT	4	9	17	32	49
		CPU	0.012	0.063	0.339	1.202	3.241
	Picard	IT	16	37	155	Fail	Fail
		CPU	0.006	0.036	0.339	-	-
	GGS	IT	Fail	89	43	163	450
		CPU	-	0.014	0.023	0.117	0.432
	PGGS	IT	25	10	13	32	90
		CPU	0.004	0.004	0.011	0.029	0.094
		$\beta_{\exp}$	0.7	1.3	1.4	1.5	1.6

Table 4: Numerical results for Example 3 for different values of n and q and $p = -1$.

q	Method	n	400	1600	3600	6400	10000
1	Picard-HSS	IT	61	235	433	748	Fail
		CPU	0.058	1.011	7.296	31.068	-
		$\alpha_{\exp}$	5.7	5.7	5.2	5.1	-
	Gen. Newton	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	Picard	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	GGS	IT	637	2425	5367	9462	Fail
		CPU	0.052	0.333	2.306	5.762	-
	PGGS	IT	55	298	666	1178	1834
		CPU	0.007	0.056	0.330	0.915	1.773
		$\beta_{\exp}$	1.6	1.5	1.5	1.5	1.5
10	Picard-HSS	IT	27	74	150	256	391
		CPU	0.036	0.339	2.701	10.499	29.236
		$\alpha_{\exp}$	6.7	5.6	5.3	5.2	5.1
	Gen. Newton	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	Picard	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	GGS	IT	198	777	1738	3080	4803
		CPU	0.013	0.102	0.745	1.980	4.342
	PGGS	IT	46	62	152	279	581
		CPU	0.004	0.015	0.076	0.209	0.561
		$\beta_{\exp}$	1.8	1.7	1.6	1.6	1.5
100	Picard-HSS	IT	20	64	Fail	Fail	Fail
		CPU	0.026	0.284	-	-	-
		$\alpha_{\exp}$	6	5.8	-	-	-
	Gen. Newton	IT	6	21	Fail	Fail	Fail
		CPU	0.028	0.167	-	-	-
	Picard	IT	18	66	Fail	Fail	Fail
		CPU	0.020	0.058	-	-	-
	GGS	IT	Fail	Fail	217	113	177
		CPU	-	-	0.040	0.077	0.175
	PGGS	IT	50	26	143	22	22
		CPU	0.007	0.005	0.074	0.020	0.027
		$\beta_{\exp}$	0.6	1	0.6	1.3	1.3

Example 4. Consider the AVE (1), in which $A \in \mathbb{R}^{n\times n}$ is given by $A = M + \mu I$, where

$$M = \begin{pmatrix} S & -I & 0 & \cdots & 0 & 0 \\ -I & S & -I & \cdots & 0 & 0 \\ 0 & -I & S & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & S & -I \\ 0 & 0 & \ddots & \ddots & -I & S \end{pmatrix} \in \mathbb{R}^{m^2\times m^2},$$

where $S = \text{tridiag}(-1, 4, -1) \in \mathbb{R}^{m\times m}$ and $I \in \mathbb{R}^{m\times m}$ is the identity matrix and $0 \in \mathbb{R}^{m\times m}$ is the zero matrix. This problem may arise from finite difference discretization on equidistant grid of a free boundary value problem about the flow of water through a porous dam [6]. Let m be a prescribed positive integer and $n = m^2$. We present the numerical results for different values of m ($m = 50, 100, 200, 300, 400$) and μ ($\mu = 0, -0.5, -0.9$) in Table 5. It can be seen that the Picard, the GGS and the generalized Newton methods fail to converge for several cases, but the PGGS converges to the solution of the problem properly. Moreover, the PGGS method has the least CPU time among the tested methods.

6 Conclusion

In this paper, we have investigated the Gauss-Seidel iteration method for solving a class of AVE and its preconditioned version. Accordingly, the GGS and the PGGS methods have been presented and sufficient conditions for the convergence of the methods have been provided. The methods are easily implemented in practice. Moreover, some numerical results have been given to show the efficiency of the methods. Numerical results showed that the proposed schemes are applicable to large and sparse AVE and the PGGS iteration method is significantly superior to other methods. Theoretical comparison and analysis of these iteration methods are interesting topics in future study.

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References

[1] O. Axelsson, *Iterative solution methods*, Cambridge University Press, Cambridge, 1996.

[2] Z.-Z. Bai, *Modulus-based matrix splitting iteration methods for linear complementarity problems*, Numer. Linear Algebra Appl. **17** (2010) 917–933.

[3] S.-J. Chung, *NP-completeness of the linear complementarity problem*, J. Optim. Theory Appl. **60** (1989) 393–399.

[4] R.W. Cottle and G. Dantzig, *Complementary pivot theory of mathematical programming*, Linear Algebra Appl. **1** (1968) 103–125.

Table 5: Numerical results for Example 4 for different values of m and μ.

μ	Method	m	50	100	200	300	400
0	Picard-HSS	IT	31	29	28	27	27
		CPU	0.152	1.309	11.392	47.423	209.097
		$\alpha_{\exp}$	7	7	7	7	7
	Gen. Newton	IT	5	5	5	5	5
		CPU	0.022	0.159	0.831	2.174	4.510
	Picard	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	GGS	IT	114	122	168	Fail	Fail
		CPU	0.035	0.119	0.524	-	-
	PGGS	IT	18	18	18	17	16
		CPU	0.006	0.023	0.088	0.197	0.364
		$\beta_{\exp}$	1.3	1.3	1.3	1.3	1.2
-0.5	Picard-HSS	IT	14	13	13	13	13
		CPU	0.075	0.606	6.333	36.483	172.715
		$\alpha_{\exp}$	9	9	9	9	9
	Gen. Newton	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	Picard	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	GGS	IT	74	77	Fail	Fail	Fail
		CPU	0.017	0.074	-	-	-
	PGGS	IT	20	20	20	20	20
		CPU	0.009	0.030	0.094	0.210	0.418
		$\beta_{\exp}$	1.1	1.1	1.1	1.1	1.1
-0.9	Picard-HSS	IT	31	32	32	32	32
		CPU	0.183	1.382	11.127	187.232	196.297
		$\alpha_{\exp}$	6.3	6.2	6.1	6.1	6.1
	Gen. Newton	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	Picard	IT	Fail	Fail	Fail	Fail	Fail
		CPU	-	-	-	-	-
	GGS	IT	283	300	Fail	Fail	Fail
		CPU	0.054	0.286	-	-	-
	PGGS	IT	56	59	60	61	61
		CPU	0.018	0.064	0.228	0.526	1.030
		$\beta_{\exp}$	1.3	1.3	1.3	1.3	1.3

[5] R.W. Cottle, J.S. Pang and R.E. Stone, *The linear complementarity problem*, SIAM, Philadelphia, (2009).

[6] C.M. Elliot and J.R. Ockenden, *Weak Variational Methods for Moving Boundary Value Problems*, Pitman, London, 1982.

[7] S. Ketabchi and H. Moosaei, *Optimal correcting and methods of feasible directions*, J. Optim. Theory Appl. **154** (2012) 209–216.

[8] S. Ketabchi, H. Moosaei and S. Fallahi *Optimal error correction of the absolute value equation using a genetic algorithm*, Mathematical and Computer Modelling, **57** (2013) 2339–2342.

[9] H. Kotakemori, H. Niki and N. Okamoto, *Accelerated iterative method for Z-matrices*, J. Comput. Appl. Math. **75** (1996) 87–97.

[10] O.L. Mangasarian, *A generalized Newton method for absolute value equations*, Optim. Lett. **3** (2009) 101–108.

[11] O.L. Mangasarian and R.R. Meyer, *Absolute value equations*, Linear Algebra Appl. **419** (2006) 359–367.

[12] K.G. Murty, *Linear complementarity, linear and nonlinear programming*, Heldermann, Berlin (1988).

[13] J. Rohn, *A theorem of the alternatives for the equation* $A\boldsymbol{x} + B|\boldsymbol{x}| = \boldsymbol{b}$, Linear Multilinear Algebra **52** (2004) 421–426.

[14] J. Rohn, *An algorithm for computing all solutions of an absolute value equation*, Optim. Lett. **6** (2012) 851–856.

[15] J. Rohn, V. Hooshyarbakhsh and R. Farhadsefat, *An iterative method for solving absolute value equations and sufficient conditions for unique solvability*, Optim. Lett. 8 (2014) 35-44.

[16] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, Springer, New York, 2002.

[17] D.K. Salkuyeh, *The Picard-HSS iteration method for absolute value equations*, Optim. Lett. **24** (2014).

[18] Y. Saad, *Iterative Methods for Sparse Linear Systems*, Second Edition, SIAM, 2003.

[19] R.S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, 1962.

[20] D. Serre, *Matrices: Theory and Applications*, Springer, New York, 2002.