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An explicit formula for the inverse of arrowhead and doubly arrow matrices

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Abstract In [Applied Mathematics Letters 33 (2014) 1-5], Najafi et al. have elaborated an approach to compute the inverse of arrowhead matrices. This paper concerns with offering an alternative simple and neat framework to obtain an explicit formula for the inverse of arrowhead matrices. More precisely, the adopted manner makes us capable to derive the inverse of block arrowhead matrices. In addition, we propound a strategy to reckon the inverse of doubly arrow matrices. Finally illustrative examples are examined which reveal the applicability and feasibly of the handled strategies.

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1 Introduction

Arrowhead matrices play a cardinal role in numerous areas, see [2, 5, 11, 16]. For instance, this family of matrices appearers when Lanczos method is utilized for solving the eigenvalue problem for large and sparse matrices [10]. Furthermore, arrow matrices occur in certain symmetric inverse eigenvalue and inverse Sturm–Liouville problems which have many applications, involving contemporary control theory and vibration analysis; for further details see [6, 12, 13] and the reference therein. Moreover, they emerges in the eigenstructure problems of arrowhead matrices which materialize from applications in molecular physics [9] and also in the implementation of the finite element or finite difference method over a region by removing part of the region [1].

In [3], adaptive approximate inverse arrow matrix techniques have been developed for arrowhead matrices based on LU-type factorization procedures without inverting the related decomposition factors. More precisely, Gravvanis stated that the main inspiration for obtaining the approximate inverse arrow matrix techniques lies in the fact that they can be employed in conjunction with explicit iterative schemes and are suitable for solving large linear systems on parallel and vector processors.

Motivated by wide applications of arrowhead matrices, Najafi et al. [8] have focused on finding an approach to invert arrowhead matrices. As a matter of fact the authors first consider the decomposition $A = I + S_1 + S_2$ for a given nonsingular arrowhead matrix A whose all of its diagonal entries are assumed to be equal to one, the matrices S_1 and S_2 are respectively stand for

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the strictly lower and strictly upper triangular parts of A. Afterward, the authors have utilized the well-known Sherman-Morrison formula to derive an expression for the inverse of arrowhead matrices. The reported numerical experiments have demonstrated the superior performance of their suggested approach in comparison with the offered manners proposed in [3,4].

The outline of this paper is organized as follows. In Section 2, we elucidate how to obtain explicit formulas for the inverse of arrowhead, block arrowhead and doubly arrow matrices, respectively. In order to turn out the effectiveness of the proposed techniques, we appraise them practically in Section 3. Finally, the paper is ended with a brief conclusion in Section 4.

2 Main results

The current section consists of two subsections. In the first part, we apply a new procedure to find explicit formulas for the inverse of an arrowhead matrix and its block version. Whereas, the second subsection is devoted to deriving the inverse of doubly arrow matrices by exploiting the results obtained in the first subsection.

The subsequent theorem is useful for obtaining our main results and its proof can be found in [7].

Theorem 1. Presume that

$$M = \begin{pmatrix} A & C \\ R & B \end{pmatrix},$$

where $A \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{s\times s}$, $C \in \mathbb{R}^{r\times s}$ and $R \in \mathbb{R}^{s\times r}$.
(a) If A and $S = B - RA^{-1}C$ are nonsingular, then

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}CS^{-1}RA^{-1} & -A^{-1}CS^{-1} \\ -S^{-1}RA^{-1} & S^{-1} \end{pmatrix}.$$

(b) If B and $T = A - CB^{-1}R$ are nonsingular, then

$$M^{-1} = \begin{pmatrix} T^{-1} & -T^{-1}CB^{-1} \\ -B^{-1}RT^{-1} & B^{-1} + B^{-1}RT^{-1}CB^{-1} \end{pmatrix}.$$

2.1 Explicit formulas for the inverses of an arrowhead matrix and its block version

Consider the nonsingular arrowhead matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ as follows:

$$A = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 & b_1 \\ 0 & d_2 & 0 & \cdots & 0 & b_2 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & d_{n-1} & 0 & b_{n-1} \\ 0 & 0 & \cdots & 0 & d_n & b_n \\ c_1 & c_2 & \cdots & c_{n-1} & c_n & \alpha \end{pmatrix}. \tag{1}$$

In this subsection, we aim to derive an explicit formula for the inverse of A. Recently, Najafi et al. [8] have presented an expression for the inverse of (1). In this paper we apply another strategy to derive the inverse of (1). Let us assume that

$$D = \mathrm{diag}(d_1, d_2, \ldots, d_n), \quad c = (c_1, c_2, \ldots, c_n), \quad b = (b_1, b_2, \ldots, b_n)^T. \tag{2}$$

Consequently, the matrix (1) can be partitioned as follows:

$$A = \begin{pmatrix} D & b \\ c & \alpha \end{pmatrix}.$$

According to part (a) of Theorem 1, if $\det(D) \neq 0$ and

$$\beta = \alpha - cD^{-1}b = \alpha - \sum_{i=1}^{n} \frac{b_i c_i}{d_i} \neq 0,$$

then the inverse of the arrowhead matrix (1) is given by

$$A^{-1} = \begin{pmatrix} D^{-1} + \frac{1}{\beta} D^{-1} b c D^{-1} & -\frac{1}{\beta} D^{-1} b \\ -\frac{1}{\beta} c D^{-1} & \frac{1}{\beta} \end{pmatrix}. \tag{3}$$

From Eq. (3), the entries of A^{-1} can explicitly written in the ensuing form:

$$(A^{-1})_{ij} = \begin{cases} \frac{1}{d_i} + \frac{1}{\beta} \frac{b_i c_i}{d_i^2}, & 1 \leq i = j \leq n, \\ \frac{1}{\beta} \frac{b_i c_j}{d_i d_j}, & 1 \leq i \neq j \leq n, \\ -\frac{1}{\beta} \frac{c_j}{d_j}, & i = n+1, \quad j = 1, \ldots, n, \\ -\frac{1}{\beta} \frac{b_i}{d_i}, & i = 1, \ldots, n, \quad j = n+1, \\ \frac{1}{\beta}, & i = j = n+1. \end{cases} \tag{4}$$

For simplicity, we set $\tilde{b} = D^{-1}b$, $\tilde{c} = cD^{-1}$, $\tilde{\beta} = 1/\beta$ and $\tilde{d}_i = 1/d_i$ for $i = 1, 2, \ldots, n$. Therefore, in practice the entries of A^{-1} are computed by Algorithm 1.

Algorithm 1. ***ArrInv***

1. Input D, c and b defined in Eq. (2) and α.
2. Set $\tilde{d}_i = 1/d_i$, $\tilde{c}_i = c_i/d_i$ and $\tilde{b}_i = b_i/d_i$ for $i = 1, 2, \ldots, n$.
3. Compute $\beta = \alpha - \sum_{i=1}^{n} b_i \tilde{c}_i$, and $\tau = 1/\beta$.
4. Set $g_{ii} = \tilde{d}_i + \tau \tilde{b}_i \tilde{c}_i$ for $i = 1, \ldots, n$, and $g_{ij} = \tau \tilde{b}_i \tilde{c}_j$ for $1 \leq i \neq j \leq n$.
5. Set $g_{n+1,j} = -\tau \tilde{c}_j$ for $j = 1, \ldots, n$ and $g_{i,n+1} = -\tau \tilde{b}_i$ for $i = 1, \ldots, n$.
6. Set $g_{n+1,n+1} = \tau$
7. Set $A^{-1} = (g_{ij})$

An analogues manner can be implemented to gain an explicit formula for the subsequent nonsingular block arrowhead matrix,

$$A = \begin{pmatrix} D_1 & 0 & 0 & \cdots & 0 & B_1 \\ 0 & D_2 & 0 & \cdots & 0 & B_2 \\ 0 & 0 & D_3 & \cdots & 0 & B_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_n & B_n \\ C_1 & C_2 & C_3 & \cdots & C_n & \Omega \end{pmatrix}.$$

Suppose that the matrices D_i are nonsingular for $i = 1, 2, \ldots, n$ and $\det(J) \neq 0$ where

$$J = \Omega - \sum_{i=1}^{n} C_i D_i^{-1} B_i.$$

In this case, it is not onerous to see that if $G = A^{-1} = (G_{ij})$ with G_{ij}'s having appropriate sizes then

$$G_{ij} = \begin{cases} D_i^{-1} + D_i^{-1}B_iJ^{-1}C_iD_i^{-1}, & 1 \le i = j \le n, \\ D_i^{-1}B_iJ^{-1}C_jD_j^{-1}, & 1 \le i \ne j \le n, \\ -J^{-1}C_jD_j^{-1}, & i = n+1, \quad j = 1, \ldots, n, \\ -D_i^{-1}B_iJ^{-1}, & i = 1, \ldots, n, \quad j = n+1, \\ J^{-1}, & i = j = n+1. \end{cases} \tag{5}$$

2.2 An explicit formula for the inverse of doubly arrow matrices

Consider the following doubly arrow matrix,

$$\mathcal{A} = \begin{pmatrix} d_1 & 0 & \cdots & 0 & b_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 & b_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n & b_n & 0 & 0 & \cdots & 0 \\ c_1 & c_2 & \cdots & c_n & \alpha & q_1 & q_2 & \cdots & q_n \\ 0 & 0 & \cdots & 0 & p_1 & f_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & p_2 & 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & p_n & 0 & 0 & \cdots & f_n \end{pmatrix}, \tag{6}$$

which was firstly considered by Pickmann et al. [14]. The matrix $\mathcal{A}$ consists of two arrow matrices, forward and backward. In fact this class of matrices incorporates the matrices of the form (1) as a special instance. Nevertheless, to the best of our knowledge, the problem of finding the inverse of this type of matrices has not been investigated so far. Utilizing the results obtained in the previous subsection, we elaborate an explicit formula for the inverse of $\mathcal{A}$.

Let us define

$$F = \mathrm{diag}(f_1, f_2, \ldots, f_n), \quad p = (p_1, p_2, \ldots, p_n)^T, \quad q = (q_1, q_2, \ldots, q_n). \tag{7}$$

Then the matrix $\mathcal{A}$ can be partitioned as follows:

$$\mathcal{A} = \begin{pmatrix} A & W \\ V & F \end{pmatrix},$$

where A is specified by (1), $V = pe_{n+1}^T$ and $W = e_{n+1}q$ in which e_{n+1} is last column of the identity matrix of order $n+1$. In view of Theorem 1, if the matrices F and $T = A - WF^{-1}V$ are nonsingular then the matrix (6) is invertible and its inverse is given by

$$\mathcal{A}^{-1} = \begin{pmatrix} T^{-1} & -T^{-1}WF^{-1} \\ -F^{-1}VT^{-1} & F^{-1} + F^{-1}VT^{-1}WF^{-1} \end{pmatrix}.$$

To compute the inverse of $\mathcal{A}$ explicitly, we need to compute each block of $\mathcal{A}^{-1}$. First of all, we have

$$T = A - WF^{-1}V = A - e_{n+1}(qF^{-1}p)e_{n+1}^T = \begin{pmatrix} D & b \\ c & \gamma \end{pmatrix}, \tag{8}$$

where

$$\gamma = \alpha - \sum_{i=1}^{n} \frac{p_iq_i}{f_i}.$$

This reveals that T is itself an arrowhead matrix and its inverse can be computed by Algorithm 1. The (2,2)-block of $\mathcal{A}^{-1}$ is computed by

$$\begin{aligned}\left(F^{-1}+F^{-1}VT^{-1}WF^{-1}\right)_{ij} &= \left(F^{-1}+F^{-1}pe_{n+1}^{T}T^{-1}e_{n+1}qF^{-1}\right)_{ij}\\ &= \left(F^{-1}+\eta F^{-1}pqF^{-1}\right)_{ij}\\ &= \begin{cases}\dfrac{1}{f_i}+\eta\dfrac{p_iq_i}{f_i^2}, & 1\le i=j\le n,\\ \eta\dfrac{p_iq_j}{f_if_j}, & 1\le i\ne j\le n,\end{cases}\end{aligned}$$

where

$$\eta=(T^{-1})_{n+1,n+1}=\frac{1}{\gamma-\sum_{i=1}^{n}\dfrac{b_ic_i}{d_i}}.$$

Applying the same manners, we have

$$(-F^{-1}VT^{-1})_{ij}=(-F^{-1}pe_{n+1}^{T}T^{-1})_{ij}=-\frac{p_i}{f_i}(T^{-1})_{n+1,j},\quad i=1,\dots,n, j=1,\dots,n+1,$$

and

$$(-T^{-1}WF^{-1})_{ij}=(-T^{-1}e_{n+1}qF^{-1})_{ij}=-\frac{q_j}{f_j}(T^{-1})_{i,n+1},\quad i=1,\dots,n+1, j=1,\dots,n.$$

Analogous to the ArrInv algorithm we present Algorithm 2 which computes the inverse of a doubly arrow matrix.

Algorithm 2. ***DArrInv***

1. Input D, c and b defined in Eq. (2), F, p and q defined in Eq. (7) and α.
2. Set $\tilde{d}_i=1/d_i$, $\tilde{c}_i=c_i/d_i$, $\tilde{b}_i=b_i/d_i$, $\tilde{f}_i=1/f_i$, $\tilde{p}_i=p_i/f_i$ and $\tilde{q}_i=q_i/f_i$, for $i=1,2,\dots,n$.
3. Set $\gamma=\alpha-\sum_{i=1}^{n}p_i\tilde{q}_i$, and compute $H=(h_{ij})=T^{-1}$ by Algorithm 1, where T is defined by Eq. (8). Set $g_{ij}=h_{ij}$, for $i,j=1,\dots,n+1$.
4. Compute $\eta=1/(\gamma-\sum_{i=1}^{n}b_i\tilde{c}_i)$
5. Set $g_{n+1+i,n+1+i}=\tilde{f}_i+\eta\tilde{p}_i\tilde{q}_i$, $i=1,\dots,n$, and $g_{n+1+i,n+1+j}=\eta\tilde{p}_i\tilde{q}_j$, $1\le i\ne j\le n$,
6. Set $g_{n+1+i,j}=-\tilde{p}_ih_{n+1,j}$ for $i=1,\dots,n$, $j=1,\dots,n+1$.
7. Set $g_{i,n+1+j}=-\tilde{q}_ih_{i,n+1}$ for $i=1,\dots,n+1$, $j=1,\dots,n$.
8. Set $\mathcal{A}^{-1}=(g_{ij})$

3 Illustrative examples

In this section we scrutinize the offered strategies for two examples to embellish the efficiency of the DArrInv algorithm.

Example 1. Consider the matrix $\mathcal{A}$ defined by (6) with $d_i=f_i=1$, $b_i=c_i=s$, $p_i=q_i=t$ for $i=1,2,\dots,n$. First of all, we have

$$\gamma=\alpha-\sum_{i=1}^{n}p_i\tilde{q}_i=\alpha-\sum_{i=1}^{n}s^2=\alpha-ns^2.$$

Using the ArrInv algorithm, we derive that

$$T^{-1} = \begin{pmatrix} 1+\tau t^2 & \tau t^2 & \cdots & \tau t^2 & -\tau t \\ \tau t^2 & 1+\tau t^2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \tau t^2 & -\tau t \\ \tau t^2 & \cdots & \tau t^2 & 1+\tau t^2 & -\tau t \\ -\tau t & \cdots & -\tau t & -\tau t & \tau \end{pmatrix},$$

where

$$\tau = \frac{1}{\gamma - \sum_{i=1}^{n} b_i \tilde{c}_i} = \frac{1}{\alpha - ns^2 - \sum_{i=1}^{n} t^2} = \frac{1}{\alpha - n(s^2+t^2)}.$$

Obviously, we have $\eta = \tau$. Therefore, from Steps 5, 6 and 7 of the DArrInv algorithm we would have

$$\mathcal{A}^{-1} = \begin{pmatrix} 1+\tau t^2 & \tau t^2 & \cdots & \tau t^2 & -\tau t & \tau ts & \cdots & \cdots & \tau ts \\ \tau t^2 & 1+\tau t^2 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \tau t^2 & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tau t^2 & \cdots & \tau t^2 & 1+\tau t^2 & -\tau t & \tau ts & \cdots & \cdots & \tau ts \\ -\tau t & \cdots & -\tau t & -\tau t & \tau & -\tau s & \cdots & \cdots & -\tau s \\ \tau ts & \cdots & \tau ts & \tau ts & -\tau s & 1+\tau s^2 & \tau s^2 & \cdots & \tau s^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \tau s^2 & 1+\tau s^2 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \tau s^2 \\ \tau ts & \cdots & \tau ts & \tau ts & -\tau s & \tau s^2 & \cdots & \tau s^2 & 1+\tau s^2 \end{pmatrix}.$$

Example 2. Suppose that the doubly arrow matrix $\mathcal{A}$ of the form (6) is determined such that $\alpha = 2n+1$, $d_i = f_i = 1$, $b_i = t$, $c_i = 1/t$, $p_i = 1/s$ and $q_i = s$ for $i = 1, 2, \ldots, n$, where s and t are two nonzero real numbers. First of all, we have

$$\gamma = \alpha - \sum_{i=1}^{n} p_i \tilde{q}_i = (2n+1) - \sum_{i=1}^{n} (s \times \frac{1}{s}) = n+1.$$

By the ArrInv algorithm, the inverse of T defined by (8) is given by

$$T^{-1} = \begin{pmatrix} 2 & 1 & \cdots & 1 & -t \\ 1 & 2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & -t \\ 1 & \cdots & 1 & 2 & -t \\ -\frac{1}{t} & \cdots & -\frac{1}{t} & -\frac{1}{t} & 1 \end{pmatrix}.$$

It is noted that the parameter τ defined in the ArrInv algorithm is computed as follows:

$$\tau = \frac{1}{\gamma - \sum_{i=1}^{n} b_i \tilde{c}_i} = \frac{1}{n+1 - \sum_{i=1}^{n} (t \times \frac{1}{t})} = 1.$$

On the other hand we have $\eta = \tau = 1$. Hence, from Steps 5, 6 and 7 of the DArrInv algorithm we deduce that

$$\mathcal{A}^{-1} = \begin{pmatrix} 2 & 1 & \cdots & 1 & -t & ts & \cdots & ts & ts \\ 1 & 2 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & -t & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 1 & 2 & -t & ts & \cdots & ts & ts \\ -\frac{1}{t} & \cdots & -\frac{1}{t} & -\frac{1}{t} & 1 & -s & \cdots & -s & -s \\ \frac{1}{st} & \cdots & \frac{1}{st} & \frac{1}{st} & -\frac{1}{s} & 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 & 2 & \ddots & \vdots \\ \frac{1}{st} & \vdots & \frac{1}{st} & \frac{1}{st} & -\frac{1}{s} & \vdots & \ddots & \ddots & 1 \\ \frac{1}{st} & \cdots & \frac{1}{st} & \frac{1}{st} & -\frac{1}{s} & 1 & \cdots & 1 & 2 \end{pmatrix}.$$

4 Conclusion

We have applied a new approach to expound an explicit formula for the inverse of arrowhead matrices and their block version. With the aid of established results, the inverse of doubly arrow matrices has been computed. To clarify the applicability and effectively of the presented results, we have exploited the offered approaches practically. Here we would like to comment that with a similar techniques utilized in this work, the inverse of the following arrowhead matrix and its block version are computable

$$B = \begin{pmatrix} \alpha & b_1 & b_2 & \cdots & b_n \\ c_1 & d_1 & 0 & \cdots & 0 \\ c_2 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & d_{n-1} & 0 \\ c_n & 0 & \cdots & 0 & d_n \end{pmatrix}.$$

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References

1. I. S. Duff, A. M. Erisman and J. K. Reid, Direct Methods for Sparse Matrices, Oxford, Clarendon Press, 1986.
2. D. Fisher, G. Golub, O. Hald, C. Leiva and O. Widlund, On Fourier-Toeplitz methods for separable elliptic problems, *Math. Comp.* 28 (1974), no. 126, 349-368.
3. G. A. Gravvanis, An approximate inverse matrix technique for arrowhead matrices, *Int. J. Comput. Math.* 70 (1998), no. 1, 35–45.
4. G. A. Gravvanis, Solving symmetric arrowhead and special tridiagonal linear systems by fast approximate inverse preconditioning, *J. Math. Model. Algorithms*, 1 (2002), 269–282.
5. H. Li, Z. Gao and D. Zhao, Least squares solutions of the matrix equation $AXB + CYD = E$ with the least norm for symmetric arrowhead matrices, *Appl. Math. Comput.* 226 (2014) 719–724.
6. Z. Liu, K. Wang and C. Xu, Extremal inverse eigenvalue problem for symmetric doubly arrow matrices, *J. Appl. Math. Comput.* DOI: 10.1007/s12190-013-0716-7.
7. C. D. Meyer, Matrix Analysis and Applied Linear Algebra, SIAM, 2004.
8. H. S. Najafi, S. A. Edalatpanah and G. A. Gravvanis, An efficient method for computing the inverse of arrowhead matrices, *Appl. Math. Lett.* 33 (2014) 1-5.
9. D. P. O'leary and G. Stewart, Computing the eigenvalues and eigenvectors of symmetric arrowhead matrices, *J. Comput. Phys.* 90 (1990) 497–405.
10. B. Parlett and B. Nour-Omid, The use of refined error bound when updating eigenvalues of tridiagonals, *Linear Algebra Appl.* 68 (1985) 179-219.
11. B. Parlett, The Symmetric Eigenvalue Problem, Prentice-Hall, Englewood Cilffs, 1980.
12. J. Peng, X. Y. Hu and L. Zhang, Two inverse eigenvalue problems for a special kind of matrices, *Linear Algebra Appl.* 416 (2006), no. 2, 336-347.

13. H. Pickmann, J. Egana and R. L. Soto, Extremal inverse eigenvalue problem for bordered diagonal matrices, *Linear Algebra Appl.* 427 (2007), no. 2, 256-271.
14. H. Pickmann, J. Egana and R. L. Soto, Two inverse eigenproblems for symmetric doubly arrow matrices, *Electron. J. Linear Algebra*, 18 (2009) 700–718.
15. Y. Saad, Iterative Methods for Sparse Linear Systems, SIAM, Philadelphia, 2003.
16. H. Zha, A two-way chasing scheme for reducing a symmetric arrowhead matrix to tridiagonal form, *J. Numer. Algebra Appl.* 1 (1992), no. 1, 49-57.