On the Preconditioned AOR Iterative Method for Z-Matrices

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Abstract. In this paper, considering a general class of preconditioner $P(\alpha)$, we study the convergence properties of the preconditioned AOR (PAOR) iterative methods for solving linear system of equations. It is shown that the spectral radius of the iteration matrix of the PAOR method has a monotonically decreasing property when the value of α increases.

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1. Introduction

We consider the following linear system of equations

$$Ax = b, \tag{1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is nonsingular with unit diagonal entries and $b \in \mathbb{R}^n$. A stationary iterative method for solving Eq. (1) is expressed as follows:

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \quad k = 0, 1, 2, \dots, \tag{2}$$

where the initial vector $x^{(0)}$ where $A = M - N$ is a splitting of A. It is well-known that the iterative method (2) is convergent for any initial guess $x^{(0)}$ if and only if $\rho(\mathcal{L}) < 1$ where $\rho(.)$ denotes the spectral radius of the matrix. Let $A = I - L - U$ where I, $-L$ and $-U$ are identity matrix, strictly lower and strictly upper triangular matrices, respectively. The accelerated overrelaxation (AOR) method for solving Eq. (1) is specified by

$$x^{(k+1)} = \mathcal{L}_{\gamma,\omega}x^{(k)} + \omega(I - \gamma L)^{-1}b, \qquad k = 0, 1, 2, \dots$$

in which

$$\mathcal{L}_{\gamma,\omega} = (I - \gamma L)^{-1}[(1 - \omega)I + (\omega - \gamma)L + \omega U,$$

where ω and γ are real parameters and $\omega \neq 0$ [4].

In order to accelerate the convergence rate of the AOR method, the method is applied to the preconditioned linear system $PAx = Pb$ where the nonsingular matrix P is called the

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preconditioner. In [12], the authors have investigated the properties of the preconditioners of the form

$$P = (p_{ij}) = (-\alpha_{ij} a_{ij}), \tag{3}$$

where $p_{ii} = 1$ and $0 \leq \alpha_{ij} \leq 1$ for $i, j = 1, \ldots, n$ $(i \neq j)$. In fact, the preconditioner (3) is a generalization of the several preconditioners in the literature; see for example [2, 5–8, 12, 17]

In this paper, it is shown that under some certain assumptions, among the preconditioners of the form (3), the one with $\alpha_{ij} = 1$ for $i, j = 1, 2, \ldots, n$ $(i \neq j)$ surpasses the other preconditioners. In the sequel, we state some definitions and theorems which are utilized throughout of the paper.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is said to be nonnegative and denoted by $A \geq 0$ if $a_{ij} \geq 0$ for $i, j = 1, 2, \ldots, n$. A matrix A is called positive and represented by $A \gg 0$ if all of its entries are positive. If $A \geq 0$, then the well-known Perron-Frobenius theorem implies that $\rho(A)$ is an eigenvalue of A, see [1]. In addition, corresponding to $\rho(A)$, the matrix A has a nonnegative eigenvector called a Perron vector of A. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is an *Z-matrix* if $a_{ij} \leq 0$ for $i \neq j$. A Z-matrix A is said to be an *M-matrix* if A is nonsingular and $A^{-1} \geq 0$. A real matrix A is called *monotone* if $Ax \geq 0$ implies $x \geq 0$. A matrix A is said to be reducible if there is a permutation matrix P such that PAP^T is a block upper triangular matrix. Otherwise, it is irreducible. A directed graph is a graph where the edges have a direction associated with them. A graph is called strongly connected if every vertex is reachable from every other vertex. A matrix A is irreducible if the directed graph associated to A is strongly connected [13]. Throughout this paper the directed graph of matrix A is denoted by $\mathcal{G}(A)$.

Definition 1. Suppose that the matrix $A \in \mathbb{R}^{n\times n}$ is given. The representation $A = M - N$ is called a *splitting* of A if M is nonsingular. The splitting $A = M - N$ is called
(a) *convergent* if $\rho(M^{-1}N) < 1$;
(b) *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$;
(c) an *M-splitting* of A if M is an M-matrix and $N \geq 0$.

Lemma 1. *[16, Lemma 1.6] Let A be a Z-matrix. Then, A is an M-matrix if and only if there is a positive vector x such that $Ax \gg 0$.*

Lemma 2. *[14, Theorem 2.1] Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weak regular splittings of A where $A^{-1} \geq (>)0$, if $M_1^{-1} \geq (>) M_2^{-1}$ then $\rho(M_1^{-1}N_1) \leq (<) \rho(M_2^{-1}N_2)$.*

Theorem 1. *[9, Theorem 1.29] Let B be a nonnegative matrix. The $\rho(B) < 1$ if and only if $I - B$ is nonsingular and $(I - B)^{-1}$ is nonnegative.*

The remainder of this paper is organized as follows. In Section 2, the main results are given which focuses on studying the convergence properties of the preconditiond AOR method with respect to the value of α. Finally, the paper is ended with a succinct conclusion in Section 3.

2. Main results

In this section, we examine the influence of the preconditioner $\tilde{P} := \tilde{P}(\alpha) = (\tilde{p}_{ij}) \in \mathbb{R}^{n\times n}$ on Eq. (1) with

$$\tilde{p}_{ij} = \begin{cases} -\alpha_{ij} a_{ij}, & \text{if } i \neq j, \\ 1, & \text{otherwise,} \end{cases}$$

where $\alpha_{ij} \in \mathbb{R}$ for $i \neq j$. Let us split the preconditioner $\tilde{P}$ into $\tilde{P} = I + L(\alpha) + U(\alpha)$ in which I is the identity matrix and $L(\alpha)$ and $U(\alpha)$ are strictly lower and strictly upper triangular matrices, respectively. Presume that $\tilde{A} = \tilde{P}A = (I + L(\alpha) + U(\alpha))A$ and

$$L(\alpha)U = G_1(\alpha) + E_1(\alpha) + F_1(\alpha), \quad U(\alpha)L = G_2(\alpha) + E_2(\alpha) + F_2(\alpha),$$

where $E_1(\alpha)$ and $E_2(\alpha)$ are diagonal matrices, $F_1(\alpha)$ and $F_2(\alpha)$ are strictly lower triangular matrices and $G_1(\alpha)$ and $G_2(\alpha)$ are strictly upper triangular matrices.

In this case, the matrix $\tilde{A}$ can be decomposed as $\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U}$,

$$\begin{aligned}\tilde{D} &= I - E_1(\alpha) - E_2(\alpha),\\ \tilde{L} &= L - L(\alpha) + L(\alpha)L + F_1(\alpha) + F_2(\alpha),\\ \tilde{U} &= U + G_1(\alpha) - U(\alpha) + G_2(\alpha) + U(\alpha)U.\end{aligned}$$

If the matrix $\tilde{D} - \gamma\tilde{L}$ is nonsingular, the AOR iteration matrix for solving the preconditioned system $\tilde{P}Ax = \tilde{P}b$ can be written as

$$\tilde{\mathcal{L}}_{\gamma,\omega} = (\tilde{D} - \gamma\tilde{L})^{-1}[(1-\omega)\tilde{D} + (\omega-\gamma)\tilde{L} + \omega\tilde{U}].$$

Theorem 2. *Let A be a Z-matrix and $\alpha_{ij} \in [0,1]$ for $1 \leq i \neq j \leq n$. Then, A is an M-matrix if and only if $\tilde{A}$ is and M-matrix.*

Proof. Let A be an M-matrix and $\tilde{A} = \tilde{P}A = (\tilde{a}_{ij})$. It is easy to see that

$$\tilde{a}_{ij} = \begin{cases} 1 - \displaystyle\sum_{k=1,k\neq i}^{n} \alpha_{ik}a_{ik}a_{ki}, & 1 \leq i = j \leq n,\\ a_{ij} - \displaystyle\sum_{k=1,k\neq i}^{n} \alpha_{ik}a_{ik}a_{kj}, & 1 \leq i \neq j \leq n.\end{cases} \tag{4}$$

Since A is a Z-matrix, we have $\tilde{a}_{ij} \leq 0$ for $i,j = 1,2,\ldots,n$ and $i \neq j$. Therefore, we conclude that $\tilde{A}$ is also an Z-matrix. By Lemma 1, there exists a positive vector x such that $Ax \gg 0$. On the other hand, we have $\tilde{A} = (I + L(\alpha) + U(\alpha))Ax \gg 0$. Invoking Lemma 1, we deduce that that $\tilde{A}$ is also an M-matrix.

Conversely, let $\tilde{A}$ be an M-matrix. Evidently, $\tilde{A}^T$ is also an M-matrix and Lemma 1 implies the existence of a positive vector x for which $\tilde{A}^Tx \gg 0$, i.e., $A^T(I + L(\alpha)^T + U(\alpha)^T)x \gg 0$. For simplicity, we set $y = (I + L(\alpha)^T + U(\alpha)^T)x$. It is not difficult to see that $y \gg 0$. Thus, Lemma 1 indicates that A^T is an M-matrix. As a result, A is an M-matrix which completes the proof. $\square$

It is not difficult to prove the following proposition by using [12, Theorem 2.6].

Proposition 1. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonsingular Z-matrix, $0 \leq \gamma \leq \omega \leq 1$, $\omega \neq 0$ and $\alpha_{ij} \in [0,1]$ for $1 \leq i \neq j \leq n$. If $\rho(\mathcal{L}_{\gamma,\omega}) < 1$, then $\rho(\tilde{\mathcal{L}}_{\gamma,\omega}) \leq \rho(\mathcal{L}_{\gamma,\omega}) < 1$.*

Consider the AOR iteration matrix of the preconditioned system $\tilde{P}(\alpha)Ax = \tilde{P}(\alpha)b$ as follows:

$$\tilde{\mathcal{L}}_{\gamma,\omega}(\alpha) = (\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha))^{-1}[(1-\omega)\tilde{D}(\alpha) + (\omega-\gamma)\tilde{L}(\alpha) + \omega\tilde{U}(\alpha)],$$

where $\tilde{P}(\alpha) = I + L(\alpha) + U(\alpha)$, $\tilde{A}(\alpha) = \tilde{P}(\alpha)A = \tilde{D}(\alpha) - \tilde{L}(\alpha) - \tilde{U}(\alpha)$ in which $\tilde{D}(\alpha)$, $\tilde{L}(\alpha)$ and $\tilde{U}(\alpha)$ are the diagonal, strictly lower and strictly upper triangular matrices, respectively.

Theorem 3. *Suppose that $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a nonsingular Z-matrix. Moreover, assume that $0 \le \gamma \le \omega \le 1$, $\omega \neq 0$ and $\alpha_{ij}^{(\kappa)} \in [0,1]$ for $i,j = 1,2,\dots,n$ $(i \neq j)$ and $\kappa = 1,2$. Assume that $\rho(\mathcal{L}_{\gamma,\omega}) < 1$ and $\alpha_{ij}^{(1)} \le \alpha_{ij}^{(2)}$ for $i,j = 1,2\dots,n$. Then*

$$\rho\left(\tilde{\mathcal{L}}_{\gamma,\omega}(\alpha^{(2)})\right) \le \rho\left(\tilde{\mathcal{L}}_{\gamma,\omega}(\alpha^{(1)})\right). \tag{5}$$

Proof. Let us consider the splittings $A = M - N$ and $\tilde{A}(\alpha^{(\kappa)}) = \tilde{M}(\alpha^{(\kappa)}) - \tilde{N}(\alpha^{(\kappa)})$ where $M = \frac{1}{\omega}(I - \gamma L)$, $N = \frac{1}{\omega}[(1-\omega)I + (\omega-\gamma)L + \omega U]$ and for $\kappa = 1,2$,

$$\begin{aligned}\tilde{M}(\alpha^{(\kappa)}) &= \frac{1}{\omega}(\tilde{D}(\alpha^{(\kappa)}) - \gamma\tilde{L}(\alpha^{(\kappa)})),\\ \tilde{N}(\alpha^{(\kappa)}) &= \frac{1}{\omega}[(1-\omega)\tilde{D}(\alpha^{(\kappa)}) + (\omega-\gamma)\tilde{L}(\alpha^{(\kappa)}) + \omega\tilde{U}(\alpha^{(\kappa)})].\end{aligned}$$

It is not difficult to verify that the convergence of the AOR method implies that matrix A is an M-matrix. Consequently, Theorem 2 shows that $\tilde{A}(\alpha^{(\kappa)})$ is also an M-matrix for $\kappa = 1,2$. Hence, we deduce that $\tilde{D}^{-1}(\alpha^{(\kappa)}) \ge 0$ and $\tilde{D}^{-1}(\alpha^{(\kappa)})\tilde{L}(\alpha^{(\kappa)}) \ge 0$. By Theorem 1, it reveals that the inverse of $\tilde{M}(\alpha^{(\kappa)})$ is a nonnegative matrix. Evidently $\tilde{M}(\alpha^{(\kappa)})$ is a Z-matrix with positive diagonal entries which implies that $\tilde{M}(\alpha^{(\kappa)})$ is an M-matrix. Thus, it is not onerous to conclude that the splittings $\tilde{A}(\alpha^{(1)}) = \tilde{M}(\alpha^{(1)}) - \tilde{N}(\alpha^{(1)})$ and $\tilde{A}(\alpha^{(2)}) = \tilde{M}(\alpha^{(2)}) - \tilde{N}(\alpha^{(2)})$ are M-splitting. Evidently,

$$\tilde{D}(\alpha^{(1)}) - \tilde{D}(\alpha^{(2)}) = E_1(\alpha^{(2)}) - E_1(\alpha^{(1)}) + E_2(\alpha^{(2)}) - E_2(\alpha^{(1)}) \ge 0,$$

which is equivalent to say that $\tilde{D}(\alpha^{(1)}) \ge \tilde{D}(\alpha^{(2)})$. Note that it can be immediately concluded that

$$(\tilde{L}(\alpha^{(1)}) - \tilde{L}(\alpha^{(2)}))_{ij} = \sum_{k=1,k\neq i}^{n} (\alpha_{ik}^{(1)} - \alpha_{ik}^{(2)})a_{ik}a_{kj} \le 0,$$

which means that $\tilde{L}(\alpha^{(1)}) \le \tilde{L}(\alpha^{(2)})$.

As seen, $\tilde{A}(\alpha^{(1)}) = \tilde{M}(\alpha^{(1)}) - \tilde{N}(\alpha^{(1)})$ and $\tilde{A}(\alpha^{(2)}) = \tilde{M}(\alpha^{(2)}) - \tilde{N}(\alpha^{(2)})$ are M-splitting which is equivalent to say that $(\tilde{D}(\alpha^{(1)}) - \gamma\tilde{L}(\alpha^{(1)}))^{-1} \ge 0$ and $(\tilde{D}(\alpha^{(1)}) - \gamma\tilde{L}(\alpha^{(1)}))^{-1} \ge 0$. Straightforward computations show that $\tilde{D}(\alpha^{(2)}) - \gamma\tilde{L}(\alpha^{(2)}) \le \tilde{D}(\alpha^{(1)}) - \gamma\tilde{L}(\alpha^{(1)})$ which insinuates

$$0 \le (\tilde{D}(\alpha^{(1)}) - \gamma\tilde{L}(\alpha^{(1)}))^{-1} \le (\tilde{D}(\alpha^{(2)}) - \gamma\tilde{L}(\alpha^{(2)}))^{-1},$$

or equivalently,

$$0 \le \left(\tilde{M}(\alpha^{(1)})\right)^{-1} \le \left(\tilde{M}(\alpha^{(2)})\right)^{-1}. \tag{6}$$

For the matrix A, we consider the following two splittings $A = M_1 - N_1 = M_2 - N_2$ such that

$$M_\kappa = \left(\tilde{P}(\alpha^{(\kappa)})\right)^{-1}\tilde{M}(\alpha^{(\kappa)}) \quad \text{and} \quad N_\kappa = \left(\tilde{P}(\alpha^{(\kappa)})\right)^{-1}\tilde{N}(\alpha^{(\kappa)}),$$

where $\tilde{P}(\alpha^{(\kappa)}) = I + L(\alpha^{(\kappa)}) + U(\alpha^{(\kappa)})$ and $\kappa = 1,2$. Since $\tilde{P}(\alpha^{(2)}) \ge \tilde{P}(\alpha^{(1)}) \ge 0$, we have

$$M_2^{-1} = \left(\left(\tilde{P}(\alpha^{(2)})\right)^{-1}\tilde{M}(\alpha^{(2)})\right)^{-1} = \left(\tilde{M}(\alpha^{(2)})\right)^{-1}\tilde{P}(\alpha^{(2)}) \ge \left(\tilde{M}(\alpha^{(2)})\right)^{-1}\tilde{P}(\alpha^{(1)}).$$

By using Eq. (6) and the preceding inequality, it can be seen that

$$M_2^{-1} \geq \left(\tilde{M}(\alpha^{(1)})\right)^{-1} \tilde{P}(\alpha^{(1)}) = \left(\left(\tilde{P}(\alpha^{(1)})\right)^{-1} \tilde{M}(\alpha^{(1)})\right)^{-1} = M_1^{-1}. \tag{7}$$

It is not difficult to see that

$$\rho(M_1^{-1}N_1) = \rho\left(\left(\tilde{M}(\alpha^{(1)})\right)^{-1} \tilde{N}(\alpha^{(1)})\right) < 1,$$

and,

$$\rho(M_2^{-1}N_2) = \rho\left(\left(\tilde{M}(\alpha^{(2)})\right)^{-1} \tilde{N}(\alpha^{(2)})\right) < 1.$$

From Lemma 2, we deduce that $\rho(M_2^{-1}N_2) \leq \rho(M_1^{-1}N_1)$, which completes the proof. □

Corollary 1. *Note that Theorem 3 reveals that for improving the convergence rate of the AOR iterative method, the preconditioner $\tilde{P} = I + L + U$ is the best one between the preconditioners of the form $\tilde{P}(\alpha) = I + L(\alpha) + U(\alpha)$ with $\alpha_{ij} \in [0, 1]$.*

We would like to comment here that if A is an irreducible matrix, then $A^{-1} > 0$ [13, 15] which implies strict inequality in (5) when $\tilde{M}^{-1} < \hat{M}^{-1}$, see Lemma 2.

In the following, the set of indices (i, j) associated with the nonzero off-diagonal entries of the matrix A is represented by $\mathcal{N}_z(A)$, i.e., $\mathcal{N}_z(A) = \{(i, j)| \quad i \neq j \text{ and } a_{ij} \neq 0\}$.

Theorem 4. *Let $\mathcal{L}_{\gamma,\omega}$ and $\tilde{\mathcal{L}}_{\gamma,\omega}$ be the iteration matrices of the AOR and preconditioned AOR methods. Suppose that A is an irreducible Z-matrix, $0 \leq \gamma < 1$, $\omega \neq 0$ and $\alpha_{ij} \in [0, 1]$ for $i, j = 1, 2, \ldots, n$ $(i \neq j)$. Moreover, assume that for each $(i, j) \in \mathcal{N}_z(A)$ there exists $\tau \neq i, j$ such that $a_{ij} < \alpha_{ij}a_{ij} + \alpha_{i\tau}a_{i\tau}a_{\tau j}$. Then $\mathcal{L}_{\gamma,\omega}$ and $\tilde{\mathcal{L}}_{\gamma,\omega}$ are nonnegative and irreducible matrices.*

Proof. Let $A = I - L - U$ be an irreducible matrix, hence $\mathcal{G}(L + U)$ is strongly connected. We first show that $\tilde{A} = (\tilde{a}_{ij})$ is irreducible. To this end, we need to prove that $\mathcal{G}(\tilde{A})$ is strongly connected. Or equivalently, it is sufficient to demonstrate that $\mathcal{N}_z(A) \subseteq \mathcal{N}_z(\tilde{A})$. For $(i, j) \in \mathcal{N}_z(A)$, by the assumption, we get

$$\tilde{a}_{ij} = a_{ij} - \sum_{k=1,k\neq i}^{n} \alpha_{ik}a_{ik}a_{kj} \leq a_{ij} - \alpha_{ij}a_{ij} - \alpha_{i\tau}a_{i\tau}a_{\tau j} < 0,$$

which is equivalent to say that $(i, j) \in \mathcal{N}_z(\tilde{A})$. Thus, the matrix $\tilde{A}$ is irreducible which implies that $\mathcal{G}(\tilde{D}^{-1}(\tilde{L} + \tilde{U}))$ is strongly connected where $\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U}$.

It is not difficult to see that $\rho(\gamma\tilde{D}^{-1}\tilde{L}) < 1$. Consequently, it turns out that

$$(\tilde{D} - \gamma\tilde{L})^{-1} = \left(I + (\gamma\tilde{D}^{-1}\tilde{L}) + (\gamma\tilde{D}^{-1}\tilde{L})^2 + \cdots\right)\tilde{D}^{-1}.$$

It can be seen that if A is a Z-matrix, then $\tilde{A}$ is a Z-matrix. Therefore, $\tilde{L}, \tilde{U} \geq 0$ and we have:

$$\begin{aligned}
\tilde{\mathcal{L}}_{\gamma,\omega} &= [\, I + (\gamma\tilde{D}^{-1}\tilde{L}) + (\gamma\tilde{D}^{-1}\tilde{L})^2 + \cdots\,][(1-\omega)I + (\omega-\gamma)\tilde{D}^{-1}\tilde{L} + \omega\tilde{D}^{-1}\tilde{U}] \\
&\geq [(1-\omega)I + (\omega-\gamma)\tilde{D}^{-1}\tilde{L} + \omega\tilde{D}^{-1}\tilde{U}] + (1-\omega)(\gamma\tilde{D}^{-1}\tilde{L}) \\
&= [(1-\omega)I + \omega(1-\gamma)\tilde{D}^{-1}\tilde{L} + \omega\tilde{D}^{-1}\tilde{U}].
\end{aligned}$$

Hence, $\tilde{\mathcal{L}}_{\gamma,\omega}$ is a nonnegative and irreducible matrix. In a similar manner, we may prove that

$$\mathcal{L}_{\gamma,\omega} \geq [(1-\omega)I + \omega(1-\gamma)L + \omega U].$$

Using the facts that $L, U \geq 0$ and $\mathcal{G}(L+U)$ is strongly connected, the above relation signifies that $\mathcal{L}_{\gamma,\omega}$ is a nonnegative and irreducible matrix. □

3. Conclusion

We have studied the performance of a general class of preconditioners for the AOR iterative method. More precisely, it has been indicated that for improving the convergence rate of the AOR iterative method, the preconditioner $\tilde{P} = I + L + U$ outperforms other preconditioners of the form $\tilde{P}(\alpha) = I + L(\alpha) + U(\alpha)$ with $\alpha_{ij} \in [0, 1]$.

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