A CYCLIC ITERATIVE APPROACH AND ITS MODIFIED VERSION TO SOLVE COUPLED SYLVESTER-TRANSPOSE MATRIX EQUATIONS

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Abstract. Recently, Tang et al. [Numer. Algorithms, 66 (2014), No. 2, 379–397] have offered a cyclic iterative method for determining the unique solution of the coupled matrix equations

$$A_iXB_i = F_i, \qquad i = 1, 2, \ldots, N.$$

Analogues to the gradient-based algorithm, the proposed algorithm relies on a fixed parameter whereas it has wider convergence region. Nevertheless, the application of the algorithm to find the centro-symmetric solution of the mentioned problem has been left as a project to be investigated and the optimal value for the fixed parameter has not been derived. In this paper, we focus on a more general class of the coupled linear matrix equations that incorporate the mentioned ones in the earlier refereed work. More precisely, we first develop the authors' propounded algorithm to resolve our considered coupled linear matrix equations over centro-symmetric matrices. Afterwards, we disregard the restriction of the existence of the unique (centro-symmetric) solution and also modify the authors' algorithm by applying an oblique projection technique which allows to produce a sequence of approximate solutions which gratify an optimality property. Numerical results are reported to confirm the validity of the established results and to demonstrate the superior performance of the modified version of the cyclic iterative algorithm.

Keywords: Cyclic iterative method; Matrix equations; Centro-symmetric matrix; Oblique projection technique.
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1. Introduction

Throughout this paper we exploit $\mathrm{tr}(A), A^T, \overline{A}$ and A^H to denote the trace, the transpose, the conjugate and the conjugate transpose of a given matrix A, respectively. The notation $\mathbb{R}^{m\times n}$ stands for the set of all $m \times n$ real matrices. Assume that $Y, Z \in \mathbb{R}^{n\times p}$ are two given matrices, the inner product of Y and Z is defined by $\langle Y, Z\rangle_F = \mathrm{tr}(Y^TZ)$. The induced norm is the well-known Frobenius norm, i.e., the norm of $Y \in \mathbb{R}^{n\times p}$ is given by $\|Y\|_F = \sqrt{\mathrm{tr}(Y^TY)}$. As a natural way, the inner product of $X = (X_1, X_2, \ldots, X_q)$ and $Y = (Y_1, Y_2, \ldots, Y_q)$ can be expounded by

$$\langle X, Y\rangle = \langle X_1, Y_1\rangle_F + \langle X_2, Y_2\rangle_F + \cdots + \langle X_q, Y_q\rangle_F,$$

where $X_j, Y_j \in \mathbb{R}^{n_j\times m_j}$ for $j = 1, 2, \ldots, q$. Consequently, we may define the norm of $X = (X_1, X_2, \ldots, X_q)$ as follows:

$$\|X\|^2 = \|X_1\|_F^2 + \|X_2\|_F^2 + \cdots + \|X_q\|_F^2,$$

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where $X_j \in \mathbb{R}^{n_j \times m_j}$ for $j = 1, 2, \ldots, q$. An $n \times n$ real matrix P is said to be a reflection matrix if $P = P^T = P^{-1}$ and the set of all $n \times n$ reflection matrices is denoted by $\mathbb{SOR}^{n\times n}$. A matrix $X \in \mathbb{R}^{m\times n}$ is called a *centro-symmetric* matrix with respect to $P \in \mathbb{SOR}^{m\times m}$ and $Q \in \mathbb{SOR}^{n\times n}$ if $X = PXQ$. The symbol $\mathbb{CSR}^{m\times n}(P, Q)$ refers to the set of all $m \times n$ centro-symmetric matrices with respect to the given reflection matrices P and Q. Note that an arbitrary matrix $Z \in \mathbb{R}^{m\times n}$ is a centro-symmetric matrix with respect I_m and I_n where I_m (I_n) represents the identity matrix of order m (n). Presume that $X = (X_1, X_2, \ldots, X_q)$, we call X as a *matrix group*. The matrix group $X = (X_1, X_2, \ldots, X_q)$ is said to be a *centro-symmetric matrix group* if the matrices $X_1, X_2, \ldots, X_q$ are centro-symmetric matrices. For two integers m and n, $I[m, n]$ is used to denote the set $\{m, m + 1, \ldots, n\}$.

Iterative algorithms, such as the gradient-based method, can be exploited for estimating the parameters of systems from input-output data and have wide applications in state estimation. For example, based on the gradient search and least-squares principles, Ding et al. [12] have proposed a gradient-based and a least-squares-based iterative estimation algorithms to roughly calculate the parameters for a multi-input multi-output (MIMO) system with coloured autoregressive moving average (ARMA) noise from input-output data. In [13], a least-squares-based iterative algorithm and a gradient-based iterative algorithm have propounded for Hammerstein systems using the decomposition-based hierarchical identification principle. Recently, Xiong et al. [20] have offered a gradient-based iterative estimation algorithm to approximate the parameters of a class of Wiener nonlinear systems from input-output measurement data. In [21], the authors have developed a gradient-based iterative algorithm for the multiple-input single–output (MISO) Wiener nonlinear system.

Linear matrix equations materialize in numerous areas such as control and system theory, image processing and some other fields of applied mathematics. Before stating the new contribution of the current paper, we briefly recollect some of the recently presented papers in the subject of linear matrix equations. Hitherto, the gradient-based iterative algorithms have been wildly examined for solving different kinds of (coupled) matrix equations in the literature. For instance, Ding and Chen [4–9] have presented various iterative methods based on the hierarchical identification principle to resolve several kinds of matrix equations. In [3], Dehghan and Hajarian have offered two gradient-based algorithms for solving the following matrix equation

$$\sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{q} C_j Y D_j = F, \tag{1.1}$$

over reflexive and anti-reflexive matrices.

Recently, Ding et al. [11] have presented an iterative algorithm to resolve the coupled matrix equations $A_i X B_i = F_i$ for $i = 1, \ldots, p$. In [10], a gradient-based iterative algorithm has been suggested for solving $AXB + CXD = F$ where $A, C \in \mathbb{R}^{m\times m}$ and $B, D \in \mathbb{R}^{n\times n}$. In [14], Li and Wang have generalized the iterative method proposed in [10] to solve the following linear matrix equation

$$\sum_{i=1}^{r} A_i X B_i = C,$$

where $A_i \in \mathbb{R}^{p\times m}$, $B_i \in \mathbb{R}^{n\times q}$ for $i \in I[1, r]$.

Song et al. [18] have considered the following coupled Sylvester-transpose matrix equations

$$\sum_{\eta=1}^{p} \left(A_{i\eta} X_\eta B_{i\eta} + C_{i\eta} X_\eta^T D_{i\eta} \right) = F_i, \qquad i = 1, 2, \ldots, N,$$

where $A_{i\eta} \in \mathbb{R}^{m_i \times l_\eta}$, $B_{i\eta} \in \mathbb{R}^{n_\eta \times p_i}$, $C_{i\eta} \in \mathbb{R}^{m_i \times n_\eta}$, $D_{i\eta} \in \mathbb{R}^{l_\eta \times p_i}$, $F_i \in \mathbb{R}^{m_i \times p_i}$, for $i = 1, \ldots, N$ and $\eta = 1, \ldots, p$, are given matrices and the matrices $X_\eta \in \mathbb{R}^{l_\eta \times n_\eta}$, $\eta \in I[1,p]$, are unknown. Under the assumption that the mentioned coupled matrix equations have a unique solution, a gradient-based iterative algorithm has been proposed.

Beik et al. [1] have examined a gradient-based iterative algorithm to determine a unique reflexive (anti-reflexive) solution group of the generalized coupled Sylvester-transpose and conjugate matrix equations

$$\mathcal{T}_\nu(X) = F_\nu, \qquad \nu = 1, 2, \ldots, N,$$

where $X = (X_1, X_2, \ldots, X_p)$ is a group of unknown matrices and for $\nu \in I[1,N]$,

$$\mathcal{T}_\nu(X) = \sum_{i=1}^{p}\sum_{\mu=1}^{s_1} A_{\nu i\mu} X_i B_{\nu i\mu} + \sum_{\mu=1}^{s_2} C_{\nu i\mu} X_i^T D_{\nu i\mu} + \sum_{\mu=1}^{s_3} M_{\nu i\mu} \overline{X}_i N_{\nu i\mu} + \sum_{\mu=1}^{s_4} H_{\nu i\mu} X_i^H G_{\nu i\mu},$$

in which $A_{\nu i\mu}, B_{\nu i\mu}, C_{\nu i\mu}, D_{\nu i\mu}, M_{\nu i\mu}, N_{\nu i\mu}, H_{\nu i\mu}, G_{\nu i\mu}$ and F_ν are given matrices with suitable dimensions defined over field of complex numbers.

In the above cited works and their closely related references, the convergence of the gradient-based algorithm for solving (coupled) matrix equations has been studied under the restriction that the mentioned main problem has a unique solution. However, more recently, Salkuyeh and Beik [16] have focused on the following coupled linear matrix equations

$$\sum_{j=1}^{q} A_{ij} X_j B_{ij} = C_i, \qquad i = 1, \ldots, p.$$

and demonstrated that the hypothesis of the existence of a unique solution can be omitted. In fact, the semi-convergence of the gradient-based iterative algorithm has been established for solving the considered coupled linear matrix equations. In addition, the best convergence factor for the algorithm has been derived.

1.1. **Motivations and highlight points.** The subsequent coupled linear matrix equations have been mentioned in [19],

$$A_i X B_i = F_i \qquad i = 1, 2, \ldots, N, \tag{1.2}$$

where $A_i \in \mathbb{R}^{p_i \times m}$, $B_i \in \mathbb{R}^{n \times q_i}$ and $F_i \in \mathbb{R}^{p_i \times q_i}$ for $i \in I[1,N]$ and $X \in \mathbb{R}^{m \times n}$ is the unknown matrix to be solved. Based on the incremental subgradient method [2, 15], a cyclic iterative algorithm has been introduced which has a wider convergence region than the gradient-based iterative algorithms proposed in the literature. However the following comments on [19] inspire us for presenting the current work.

- Using the propounded algorithm to find the unique centro-symmetric solution of the linear matrix equation (1.2) has been left as a project to be undertaken.
- The proposed cyclic iterative algorithm for solving (1.2) relies on a fixed parameter and the problem of determining optimum value for the parameter has not been discussed.
- All of the established results were derived under the restriction that (1.2) has a unique solution.

In order to derive our results for more general cases, we consider the following coupled linear matrix equations which incorporate (1.2) and several previously investigated (coupled) linear matrix equations,

$$\sum_{j=1}^{q} A_{ij} X_j B_{ij} + C_{ij} X_j^T D_{ij} = F_i, \qquad i = 1, 2, \ldots, N, \tag{1.3}$$

where the matrices $A_{ij} \in \mathbb{R}^{r_i \times n_j}$, $B_{ij} \in \mathbb{R}^{m_j \times s_i}$, $C_{ij} \in \mathbb{R}^{r_i \times m_j}$, $D_{ij} \in \mathbb{R}^{n_j \times s_i}$ and $F_i \in \mathbb{R}^{r_i \times s_i}$ are given. For simplicity, we exploit the linear operator

$$\begin{aligned} \mathcal{A} : \mathbb{R}^{n_1 \times m_1} \times \ldots \times \mathbb{R}^{n_q \times m_q} &\rightarrow \mathbb{R}^{r_1 \times s_1} \times \ldots \times \mathbb{R}^{r_N \times s_N} \\ X = (X_1, X_2, \ldots, X_q) &\mapsto \mathcal{A}(X) = (\mathcal{A}_1(X), \mathcal{A}_2(X), \ldots, \mathcal{A}_N(X)) \end{aligned}$$

where $\mathcal{A}_i(X) = \sum_{j=1}^{q} A_{ij} X_j B_{ij} + C_{ij} X_j^T D_{ij}$ for $i \in I[1, N]$. Therefore, (1.3) can be rewritten as follows:

$$\mathcal{A}(X) = F, \tag{1.4}$$

where $X = (X_1, X_2, \ldots, X_q)$ and $F = (F_1, F_2, \ldots, F_N)$.

The reminder of this paper is organized as follows. In Section 2, a cyclic iterative algorithm to find the unique centro-symmetric solution group of (1.4) is proposed and its convergence is established. In Section 3, a modified version of the propounded algorithm is presented and the restriction of the uniqueness of the centro-symmetric solution group is relaxed. Numerical results are provided in Section 4 which illustrate the validity and applicability of the offered algorithm and its modified version. Finally the paper is ended with a brief conclusion in Section 5.

2. Cyclic gradient iterative algorithm

In this section we develop the cyclic iterative method presented in [19] for solving (1.4) over centro-symmetric matrices. We presume that the coupled linear matrix equations (1.4) have a unique centro-symmetric solution group $(X_1, X_2, \ldots, X_q)$ where $X_j \in \mathbb{CSR}^{n_j \times m_j}(P_j, Q_j)$ and P_j, Q_j are given reflection matrices for $j = 1, 2, \ldots, q$. The convergence region of the algorithm is established and it can be easily checked out it is wider than the convergence region of the gradient-based iterative algorithm handled in [18] for solving (1.4).

Algorithm 1. Cyclic gradient iterative algorithm

- *Input the reflection matrices* $P_j \in \mathbb{SOR}^{n_j \times n_j}$ *and* $Q_j \in \mathbb{SOR}^{m_j \times m_j}$ *for* $j \in I[1, q]$. *Choose an arbitrary initial matrix group* $X(0) = (X_1(0), X_2(0), \ldots, X_q(0))$ *such that* $X_j(0) \in \mathbb{CSR}^{n_j \times m_j}(P_j, Q_j)$ *for* $j \in I[1, q]$; *for instance* $X_1(0) = X_2(0) = \cdots = X_q(0) = 0$.
- *Compute*

$$R_{[k]} = F_{[k]} - \left(\sum_{j=1}^{q} A_{[k]j} X_j(k-1) B_{[k]j} + C_{[k]j} X_j(k-1)^T D_{[k]j} \right),$$

and

$$\begin{aligned} X_j(k) \;=\; & X_j(k-1) + \frac{\mu}{2} \Big(A_{[k]j}^T R_{[k]} B_{[k]j}^T + D_{[k]j} R_{[k]}^T C_{[k]j} \\ & + P_j (A_{[k]j}^T R_{[k]} B_{[k]j}^T) Q_j + P_j (D_{[k]j} R_{[k]}^T C_{[k]j}) Q_j \Big). \end{aligned}$$

where $[k] = (k \bmod N)$ *which takes values in* $\{1, 2, \ldots, N\}$ *and* $\mu \in \left(0, \frac{2}{L}\right)$ *where*

$$L = \max_{1 \le i \le n} \left\{ \sum_{j=1}^{q} \|A_{ij}\|_F^2 \|B_{ij}\|_F^2 + \|C_{ij}\|_F^2 \|D_{ij}\|_F^2 \right\}. \tag{2.1}$$

The following theorem supplies a sufficient condition under which the above algorithm is convergent.

Theorem 2.1. *Assume that (1.3) has a unique solution $X^* = (X_1^*, X_2^*, \ldots, X_q^*)$ such that $X_j^* \in \mathbb{CSR}^{n_j\times m_j}(P_j, Q_j)$ where the matrices $P_j \in \mathbb{SOR}^{n_j\times n_j}$ and $Q_j \in \mathbb{SOR}^{m_j\times m_j}$ are given for $j \in I[1, q]$. Suppose that $\mu \in \left(0, \frac{2}{L}\right)$ where L is given by (2.1). Then the sequence of approximate solutions $\{X(k)\}_{k=1}^{\infty}$ produced by Algorithm 1 converges to X^* for any initial guess $X(0) = (X_1(0), X_2(0), \ldots, X_q(0))$ such that $X_j(0) \in \mathbb{CSR}^{n_j\times m_j}(P_j, Q_j)$ for $j \in I[1, q]$.*

Proof. In Algorithm 1, the kth approximate solution group $X(k) = (X_1(k), X_2(k), \ldots, X_q(k))$ is constructed so that

$$\begin{aligned} X_j(k) \,=\, & X_j(k-1) + \frac{\mu}{2}\Big(A_{[k]j}^T R_{[k]} B_{[k]j}^T + D_{[k]j} R_{[k]}^T C_{[k]j} \\ & + P_j(A_{[k]j}^T R_{[k]} B_{[k]j}^T) Q_j + P_j(D_{[k]j} R_{[k]}^T C_{[k]j}) Q_j\Big), \quad j = 1, 2, \ldots, q. \end{aligned}$$

Since $P_j^2 = I_{n_j}$ and $Q_j^2 = I_{m_j}$, we may conclude that if $X_j(k-1) \in \mathbb{CSR}^{n_j\times m_j}(P_j, Q_j)$ then $X_j(k) \in \mathbb{CSR}^{n_j\times m_j}(P_j, Q_j)$ for $j \in I[1, q]$. By the assumption, we have $X_j(0) \in \mathbb{CSR}^{n_j\times m_j}(P_j, Q_j)$ for $j \in I[1, q]$. Hence, it can be verified that at each step, $X(k)$ is a group of centro-symmetric matrices, i.e., $X_j(k) \in \mathbb{CSR}^{n_j\times m_j}(P_j, Q_j)$ for $k = 1, 2, \ldots$ and $j \in I[1, q]$.

In what follows we set $\tilde{X}(k) = X(k) - X^*$, that is $\tilde{X}_j(k) = X_j(k) - X_j^*$ for $j \in I[1, q]$. It is not difficult to see that

$$\begin{aligned} R_{[k]} \,&=\, F_{[k]} - \left(\sum_{j=1}^{q} A_{[k]j} X_j(k-1) B_{[k]j} + C_{[k]j} X_j(k-1)^T D_{[k]j}\right) \\ &= -\left(\sum_{j=1}^{q} A_{[k]j} \tilde{X}_j(k-1) B_{[k]j} + C_{[k]j} \tilde{X}_j(k-1)^T D_{[k]j}\right). \end{aligned}$$

Now we may deduce that

$$\begin{aligned} \tilde{X}_j(k) \,=\, & \tilde{X}_j(k-1) - \frac{\mu}{2}\Bigg\{ A_{[k]j}^T \left(\sum_{j=1}^{q} A_{[k]j} \tilde{X}_j(k-1) B_{[k]j} + C_{[k]j} \tilde{X}_j(k-1)^T D_{[k]j}\right) B_{[k]j}^T \\ & + D_{[k]j}\left(\sum_{j=1}^{q} A_{[k]j} \tilde{X}_j(k-1) B_{[k]j} + C_{[k]j} \tilde{X}_j(k-1)^T D_{[k]j}\right)^T C_{[k]j} \\ & + P_j A_{[k]j}^T \left(\sum_{j=1}^{q} A_{[k]j} \tilde{X}_j(k-1) B_{[k]j} + C_{[k]j} \tilde{X}_j(k-1)^T D_{[k]j}\right) B_{[k]j}^T Q_j \\ & + P_j D_{[k]j}\left(\sum_{j=1}^{q} A_{[k]j} \tilde{X}_j(k-1) B_{[k]j} + C_{[k]j} \tilde{X}_j(k-1)^T D_{[k]j}\right)^T C_{[k]j} Q_j \Bigg\}. \end{aligned} \tag{2.2}$$

Therefore we derive

$$\begin{aligned} \|\tilde{X}_j(k)\|_F^2 \,=\, & \|\tilde{X}_j(k-1)\|_F^2 \\ & -\mu \operatorname{tr}\Bigg(\tilde{X}_j(k-1)^T \Bigg\{ A_{[k]j}^T \left(\sum_{j=1}^{q} A_{[k]j} \tilde{X}_j(k-1) B_{[k]j} + C_{[k]j} \tilde{X}_j(k-1)^T D_{[k]j}\right) B_{[k]j}^T \\ & + D_{[k]j}\left(\sum_{j=1}^{q} A_{[k]j} \tilde{X}_j(k-1) B_{[k]j} + C_{[k]j} \tilde{X}_j(k-1)^T D_{[k]j}\right)^T C_{[k]j} \end{aligned}$$

$$+P_jA_{[k]j}^T\left(\sum_{j=1}^{q}A_{[k]j}\tilde{X}_j(k-1)B_{[k]j}+C_{[k]j}\tilde{X}_j(k-1)^TD_{[k]j}\right)B_{[k]j}^TQ_j$$

$$\left.\left.+\ P_jD_{[k]j}\left(\sum_{j=1}^{q}A_{[k]j}\tilde{X}_j(k-1)B_{[k]j}+C_{[k]j}\tilde{X}_j(k-1)^TD_{[k]j}\right)^TC_{[k]j}Q_j\right\}\right).$$

$$+\frac{\mu^2}{4}\left\|A_{[k]j}^T\left(\sum_{j=1}^{q}A_{[k]j}\tilde{X}_j(k-1)B_{[k]j}+C_{[k]j}\tilde{X}_j(k-1)^TD_{[k]j}\right)B_{[k]j}^T\right.$$

$$+D_{[k]j}\left(\sum_{j=1}^{q}A_{[k]j}\tilde{X}_j(k-1)B_{[k]j}+C_{[k]j}\tilde{X}_j(k-1)^TD_{[k]j}\right)^TC_{[k]j}$$

$$+P_jA_{[k]j}^T\left(\sum_{j=1}^{q}A_{[k]j}\tilde{X}_j(k-1)B_{[k]j}+C_{[k]j}\tilde{X}_j(k-1)^TD_{[k]j}\right)B_{[k]j}^TQ_j$$

$$\left.+\ P_jD_{[k]j}\left(\sum_{j=1}^{q}A_{[k]j}\tilde{X}_j(k-1)B_{[k]j}+C_{[k]j}\tilde{X}_j(k-1)^TD_{[k]j}\right)^TC_{[k]j}Q_j\right\|_F^2.$$

Using the facts that $\mathrm{tr}(AB)=\mathrm{tr}(BA)$, $\mathrm{tr}(A^T)=\mathrm{tr}(A)$ and $\|PAQ\|_F=\|A\|_F$ for arbitrary given reflection matrices P and Q, we conclude that

$$\|\tilde{X}_j(k)\|_F^2\ \le\ \|\tilde{X}_j(k-1)\|_F^2$$

$$-2\mu\ \mathrm{tr}\left(A_{[k]j}\tilde{X}_j(k-1)B_{[k]j}^T\left(\sum_{j=1}^{q}A_{[k]j}\tilde{X}_j(k-1)B_{[k]j}+C_{[k]j}\tilde{X}_j(k-1)^TD_{[k]j}\right)^T\right.$$

$$\left.+C_{[k]j}\tilde{X}_j(k-1)^TD_{[k]j}\left(\sum_{j=1}^{q}A_{[k]j}\tilde{X}_j(k-1)B_{[k]j}+C_{[k]j}\tilde{X}_j(k-1)^TD_{[k]j}\right)^T\right)$$

$$+\mu^2\left\{\|A_{[k]j}\|_F^2\|B_{[k]j}\|_F^2\left\|\sum_{j=1}^{q}A_{[k]j}\tilde{X}_j(k-1)B_{[k]j}+C_{[k]j}\tilde{X}_j(k-1)^TD_{[k]j}\right\|_F^2\right.$$

$$\left.+\ \|C_{[k]j}\|_F^2\|D_{[k]j}\|_F^2\left\|\sum_{j=1}^{q}A_{[k]j}\tilde{X}_j(k-1)B_{[k]j}+C_{[k]j}\tilde{X}_j(k-1)^TD_{[k]j}\right\|_F^2\right\}.$$

Hence we conclude that

$$\sum_{j=1}^{q}\|\tilde{X}_j(k)\|_F^2\ \le\ \sum_{j=1}^{q}\|\tilde{X}_j(k-1)\|_F^2-\mu(2-\mu(\sum_{j=1}^{q}\|A_{[k]j}\|_F^2\|B_{[k]j}\|_F^2$$

$$+\ \|C_{[k]j}\|_F^2\|D_{[k]j}\|_F^2))\left\|\sum_{j=1}^{q}A_{[k]j}\tilde{X}_j(k-1)B_{[k]j}+C_{[k]j}\tilde{X}_j(k-1)^TD_{[k]j}\right\|_F^2. \tag{2.3}$$

For simplicity we suppose that

$$L_k = \sum_{j=1}^{q} \|A_{[k]j}\|_F^2 \|B_{[k]j}\|_F^2 + \|C_{[k]j}\|_F^2 \|D_{[k]j}\|_F^2.$$

Consequently (2.3) can be rewritten as follows:

$$\|\tilde{X}(k)\|^2 \leq \|\tilde{X}(k-1)\|^2 - \mu(2-\mu L_k) \left\| \sum_{j=1}^{q} A_{[k]j}\tilde{X}_j(k-1)B_{[k]j} + C_{[k]j}\tilde{X}_j(k-1)^T D_{[k]j} \right\|_F^2. \tag{2.4}$$

From the above relations it reveals that

$$\|\tilde{X}(k)\|^2 \leq \|\tilde{X}(k-1)\|^2.$$

In view of (2.4), it can be found that

$$\sum_{k=1}^{\infty} \left\| \sum_{j=1}^{q} A_{[k]j}\tilde{X}_j(k-1)B_{[k]j} + C_{[k]j}\tilde{X}_j(k-1)^T D_{[k]j} \right\|_F^2 < \infty.$$

This implies that

$$\lim_{k\to\infty} \left\| \sum_{j=1}^{q} A_{[k]j}\tilde{X}_j(k-1)B_{[k]j} + C_{[k]j}\tilde{X}_j(k-1)^T D_{[k]j} \right\|_F = 0. \tag{2.5}$$

From (2.2) we have

$$\|\tilde{X}_j(k) - \tilde{X}_j(k-1)\|_F^2 \to 0 \quad \text{as} \quad k \to \infty,$$

which shows that

$$\|\tilde{X}(k) - \tilde{X}(k-1)\|^2 \to 0 \quad \text{as} \quad k \to \infty.$$

Now using the similar approach exploited in the proof of Theorem 4.1 in [19], it is seen that

$$\|\tilde{X}_j(k+i-1) - \tilde{X}_j(k-1)\|_F^2 \to 0 \quad \text{as} \quad k \to \infty, \tag{2.6}$$

for any $i \in \{1, 2, \ldots, N\}$.

Using Eqs (2.5) and (2.6), it turns out that

$$\left\| \sum_{j=1}^{q} A_{[k+i]j}\tilde{X}_j(k-1)B_{[k+i]j} + C_{[k+i]j}\tilde{X}_j(k-1)^T D_{[k+i]j} \right\|_F \to 0 \quad \text{as} \quad k \to \infty,$$

for each $i \in \{1, 2, \ldots, N\}$ which implies that

$$\lim_{k\to\infty} \left\| \sum_{j=1}^{q} A_{\ell j}\tilde{X}_j(k-1)B_{\ell j} + C_{\ell j}\tilde{X}_j(k-1)^T D_{\ell j} \right\|_F = 0, \quad \ell = 1, 2, \ldots, N. \tag{2.7}$$

As (1.3) has a unique centro-symmetric solution group then the homogenous system

$$\sum_{j=1}^{q} A_{ij}X_jB_{ij} + C_{ij}X_j^T D_{ij} = 0, \qquad i = 1, 2, \ldots, N.$$

has a unique centro-symmetric solution group $(X_1, X_2, \ldots, X_q) = (0, 0, \ldots, 0)$. Therefore (2.7) implies that

$$\tilde{X}(k) \to 0 \quad \text{as} \quad k \to \infty,$$

which completes the proof. □

3. Implementing an oblique projection technique

In this section we assume that the coupled linear matrix equations (1.3) are consistent over centro-symmetric matrices. Here we would like to point out that in the present section, we do not impose the restriction that (1.3) has a unique centro-symmetric solution group. In order to improve the speed of convergence of Algorithm 1, we apply an oblique projection technique at each step of the algorithm. Consider the step k of the algorithm, as observed, the kth approximate solution $X(k) = (X_1(k), X_2(k), \ldots, X_q(k))$ is updated as follows:

$$X(k) = X(k-1) + \mu P_{[k]},$$

with $P_{[k]} = (P_{[k]}^1, P_{[k]}^2, \ldots, P_{[k]}^q)$ where

$$P_{[k]}^j = \frac{1}{2}\left(A_{[k]j}^T R_{[k]} B_{[k]j}^T + D_{[k]j} R_{[k]}^T C_{[k]j} + P_j (A_{[k]j}^T R_{[k]} B_{[k]j}^T) Q_j + P_j (D_{[k]j} R_{[k]}^T C_{[k]j}) Q_j\right),$$

for $j = 1, 2, \ldots, q$. Now instead of using fixed parameter μ, we determine this parameter in a progressive manner. As a matter of fact, we select μ such that

$$\left\langle \bar{R}_{[k]}, R_{[k]} \right\rangle_F = 0,$$

where $\bar{R}_{[k]} = F_{[k]} - \left(\sum_{j=1}^{q} A_{[k]j} X_j(k) B_{[k]j} + C_{[k]j} X_j(k)^T D_{[k]j}\right)$.

Hence if $\left\langle R_{[k]}, \mathcal{A}_{[k]}(P_{[k]}) \right\rangle \neq 0$ then we may derive the new approximation as follows:

$$X(k) = X(k-1) + \frac{\left\langle R_{[k]}, R_{[k]} \right\rangle}{\left\langle R_{[k]}, \mathcal{A}_{[k]}(P_{[k]}) \right\rangle} P_{[k]}.$$

in which $\mathcal{A}_{[k]}(P_{[k]}) = \sum_{j=1}^{q} A_{[k]j} P_{[k]}^j B_{[k]j} + C_{[k]j} (P_{[k]}^j)^T D_{[k]j}$ and $P_{[k]} = (P_{[k]}^1, P_{[k]}^2, \ldots, P_{[k]}^q)$.

In the following proposition we do not restrict ourself to the case that the coupled linear matrix equation (1.4) have a unique centro-symmetric solution group. That is $\mathcal{A}(X) = 0$ may have a nontrivial centro-symmetric solution group. The next proposition reveals that $\left\langle R_{[k]}, \mathcal{A}_{[k]}(P_{[k]}) \right\rangle = 0$ implies $P_{[k]} = 0$.

Proposition 3.1. *Suppose that $P_{[k]} = (P_{[k]}^1, P_{[k]}^2, \ldots, P_{[k]}^q)$ and $\mathcal{A}_{[k]}(P_{[k]})$ are defined as before. If $\left\langle R_{[k]}, \mathcal{A}_{[k]}(P_{[k]}) \right\rangle_F = 0$, then $P_{[k]} = 0$.*

Proof. Note that $P_{[k]}^j$ is a centro-symmetric matrix with respect to the reflection matrices P_j and Q_j, i.e., $P_{[k]}^j = P_j P_{[k]}^j Q_j$ for $j = 1, 2, \ldots, q$. Straightforward computations turn out

$$\begin{aligned}
\left\langle R_{[k]}, \mathcal{A}_{[k]}(P_{[k]}) \right\rangle_F = 0 &\Rightarrow \left\langle \left(\sum_{j=1}^{q} A_{[k]j} P_{[k]}^j B_{[k]j} + C_{[k]j} (P_{[k]}^j)^T D_{[k]j}\right), R_{[k]} \right\rangle_F = 0 \\
&\Rightarrow \sum_{j=1}^{q} \left\langle P_{[k]}^j, \left(A_{[k]j}^T R_{[k]} B_{[k]j}^T + D_{[k]j} R_{[k]}^T C_{[k]j}\right) \right\rangle_F = 0 \\
&\Rightarrow \sum_{j=1}^{q} \left\langle \frac{1}{2}\left(P_{[k]}^j + P_j P_{[k]}^j Q_j\right), \left(A_{[k]j}^T R_{[k]} B_{[k]j}^T + D_{[k]j} R_{[k]}^T C_{[k]j}\right) \right\rangle_F = 0 \\
&\Rightarrow \sum_{j=1}^{q} \left\langle P_{[k]}^j, P_{[k]}^j \right\rangle_F = 0 \quad \Rightarrow \quad \left\langle P_{[k]}, P_{[k]} \right\rangle = 0 \quad \Rightarrow \quad P_{[k]} = 0.
\end{aligned}$$

□

Remark 3.2. *From Proposition 3.1, we may immediately conclude that if* $\mathcal{A}(P_{[k]}) = 0$ *then* $P_{[k]} = 0$.

Now we present the following practical proposition which discloses that $P_{[k]} = 0$ implies $R_{[k]} = 0$. That is, $P_{[k]} = 0$ indicates that the current approximate solution $X(k-1)$ satisfies the $[k]$th equation of the coupled linear matrix equations (1.4).

Proposition 3.3. *Presume that the coupled linear matrix equations (1.4) are consistent over the centro-symmetric matrices and let* $\hat{X} = (\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_q)$ *be a centro-symmetric solution group of (1.4). Then*

$$\left\langle \hat{X} - X(k-1), P_{[k]} \right\rangle = \|R_{[k]}\|_F^2. \tag{3.1}$$

Proof. By some easy computations and using the fact that $\hat{X} - X(k-1)$ is a centro-symmetric matrix group, we derive

$$\begin{aligned}
\left\langle \hat{X} - X(k-1), P_{[k]} \right\rangle &= \sum_{j=1}^{q} \left\langle \hat{X}_j - X_j(k-1), \left(A_{[k]j}^T R_{[k]} B_{[k]j}^T + D_{[k]j} R_{[k]}^T C_{[k]j} \right) \right\rangle_F \\
&= \sum_{j=1}^{q} \Big\langle \Big[\left(A_{[k]j} \hat{X}_j B_{[k]j} + C_{[k]j} (\hat{X}_j)^T D_{[k]j} \right) \\
&\quad - \left(A_{[k]j} X_j(k-1) B_{[k]j} + C_{[k]j} (X_j(k-1))^T D_{[k]j} \right) \Big], R_{[k]} \Big\rangle_F \\
&= \left\langle F_{[k]} - \left(\sum_{j=1}^{q} A_{[k]j} X_j(k-1) B_{[k]j} + C_{[k]j} X_j(k-1)^T D_{[k]j} \right), R_{[k]} \right\rangle_F, \\
&= \left\langle R_{[k]}, R_{[k]} \right\rangle_F. \quad \square
\end{aligned}$$

Note that in our examined approach, at each step, we face to two different circumstances. In fact for computing the new approximation, say the kth approximate solution, we mention the following two cases

- Case I. If $P_{[k]} = 0$, then we set $X(k) = X(k-1)$.
- Case II. If $P_{[k]} \neq 0$, we compute the new approximation as follows:

$$X(k) = X(k-1) + \frac{\left\langle R_{[k]}, R_{[k]} \right\rangle}{\left\langle R_{[k]}, \mathcal{A}_{[k]}(P_{[k]}) \right\rangle} P_{[k]}.$$

Afterwards, we increase k by 1 and in the next step, again we consider Cases I and II. The computation of the approximate solutions may be continued while $\|R(k)\| \geq \epsilon$ where ϵ is a given tolerance, we comment here that an alternative stopping criterion can be also utilized.

In the next proposition, we prove that the sequence of the approximate solution obtained after employing the offered projection technique satisfies an optimality property.

Proposition 3.4. *Presume that the coupled linear matrix equations (1.4) are consistent and the centro-symmetric matrix group* $\hat{X}$ *is a solution of (1.4). Assume that* $\tilde{X} = X(k-1) + \alpha P_{[k]}$ *where* α *is a positive scalar and* $X(k-1)$ *is the* $(k-1)$*th centro-symmetric approximate solution group of (1.4). Then,*

$$\left\| \hat{X} - X(k) \right\| = \min_{\tilde{X} \in \mathcal{S}(\alpha)} \left\| \hat{X} - \tilde{X} \right\|.$$

where $\mathcal{S}(\alpha) = \left\{\tilde{X} \mid \tilde{X} = X(k-1) + \alpha P_{[k]} \text{ for some } \alpha > 0\right\}$ *and*

$$X(k) = X(k-1) + \frac{\langle R_{[k]}, R_{[k]}\rangle}{\langle R_{[k]}, \mathcal{A}_{[k]}(P_{[k]})\rangle} P_{[k]}, \qquad \langle R_{[k]}, \mathcal{A}_{[k]}(P_{[k]})\rangle \neq 0.$$

Proof. It is not onerous to see that

$$\begin{aligned}\left\langle \hat{X} - \tilde{X}, \hat{X} - \tilde{X} \right\rangle &= \left\langle \hat{X} - X(k) - (\alpha - \alpha^*)P_{[k]}, \hat{X} - X(k) - (\alpha - \alpha^*)P_{[k]} \right\rangle \\ &= \left\langle \hat{X} - X(k), \hat{X} - X(k) \right\rangle - 2(\alpha - \alpha^*) \left\langle P_{[k]}, \hat{X} - X(k) \right\rangle \\ &+ (\alpha - \alpha^*)^2 \left\langle P_{[k]}, P_{[k]} \right\rangle,\end{aligned}$$

where $\alpha^* = \frac{\langle R_{[k]}, R_{[k]}\rangle}{\langle R_{[k]}, \mathcal{A}_{[k]}(P_{[k]})\rangle}$.

On the other hand, straightforward computations show that

$$\left\langle P_{[k]}, \hat{X} - X(k) \right\rangle = \langle R_{[k]}, \bar{R}_{[k]} \rangle = 0,$$

which reveals that

$$\left\langle \hat{X} - \tilde{X}, \hat{X} - \tilde{X} \right\rangle = \left\langle \hat{X} - X(k), \hat{X} - X(k) \right\rangle + (\alpha - \alpha^*)^2 \left\langle P_{[k]}, P_{[k]} \right\rangle. \tag{3.2}$$

Consequently,

$$\|\hat{X} - X(k)\| \leq \|\hat{X} - \tilde{X}\|, \tag{3.3}$$

we would like to comment here that the above inequality holds strictly if $P_{[k]} \neq 0$. □

The following remark can be concluded from the previous proposition immediately which reveals that the cyclic iterative algorithm with projection technique is convergent to a centro-symmetric solution group of (1.4) for an arbitrary initial centro-symmetric matrix group $X(0) = (X_1(0), X_2(0), \ldots, X_q(0))$ such that $X_j(0) \in \mathbb{CSR}^{n_j \times m_j}(P_j, Q_j)$ where $P_j \in \mathbb{SOR}^{n_j \times n_j}$ and $Q_j \in \mathbb{SOR}^{m_j \times m_j}$ are given for $j = 1, 2, \ldots, q$.

Remark 3.5. *Under the same assumptions in the previous proposition, by setting $\alpha = 0$ and in view of (3.3), we may conclude that*

$$\|\hat{X} - X(k)\| \leq \|\hat{X} - X(k-1)\|,$$

where $\hat{X}$ is an arbitrary solution of (1.4). Therefore,

$$\|\hat{X} - X(k)\| \to \ell, \qquad \text{as} \quad k \to \infty.$$

Note that ℓ is not necessarily zero. Now from (3.2), we deduce that there exists a positive integer N such that

$$P_{[k]} = 0, \qquad k \geq N.$$

Thence Proposition 3.3 implies that for eventually large values of k, we have

$$R_{[k]} = 0.$$

That is, there exists an integer $N > 0$ such that $R_{[k]} = 0$ for $k \geq N$ which indicates that $X(k)$ converges to a centro-symmetric solution group of (1.4).

Remark 3.6. *Suppose that the coupled matrix equations (1.4) have infinity number of centro-symmetric solution groups. With a similar strategy used in [17] and some straightforward computations, it can be verified that the minimum norm centro-symmetric solution group of (1.4) can be obtained by setting $X(0) = (X_1(0), X_2(0), \ldots, X_q(0)) = (0, 0, \ldots, 0)$.*

4. Numerical experiments

In this section, some numerical experiments are presented to illustrate the effectiveness of the proposed algorithm and the validity of the presented theoretical results. All the numerical experiments were computed in double precision using some MATLAB codes on a PC Pentium 4, with a 3.00 GHz CPU and 3.25 GB of RAM. We utilize a zero matrix as an initial guess and the subsequent stopping criterion

$$\|X_1(k) - X_1(k-1)\|_\infty < \delta,$$

is always exploited where $X(k)$ is the computed solution at iteration k, X^* is the exact solution and $\delta > 0$ is a prescribed tolerance.

Example 4.1. In the first example, we consider the coupled linear matrix equations

$$\begin{cases} X_1 + C_{11}X_1^T D_{11} = F_1, \\ A_{21}X_1B_{21} + X_1^T = F_2, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} C_{11} &= \text{tridiag}_n(-1, 3, 1), \quad D_{11} = \text{tridiag}_n(-1, 0, -1), \\ A_{21} &= \text{tridiag}_n(1, 2, 1), \quad\quad B_{21} = \text{tridiag}_n(-1, 2, -1). \end{aligned}$$

We make the right-hand side matrices F_1 and F_2 as follows. Let $Z = \text{tridiag}_n(1, 1, 1)$ and

$$P_1 = I - 2\frac{ee^T}{e^Te}, \quad \text{and} \quad Q_1 = I - 2\frac{vv^T}{v^Tv},$$

where $e = (1, 1, \ldots, 1)^T$ and $v = (v_1, v_2, \ldots, v_n)$ with $v_i = (-1)^i$, $i = 1, 2, \ldots, n$. It is known that the Householder matrices P_1 and Q_1 are reflection matrices. Evidently, $X_1^* = Z + P_1ZQ_1$ is a centro-symmetric with respect to P_1 and Q_1. Now we assume that

$$\begin{cases} F_1 = X_1^* + C_{11}(X_1^*)^T D_{11}, \\ F_2 = A_{21}X_1^*B_{21} + (X_1^*)^T. \end{cases}$$

Therefore, it is guaranteed that the matrix X_1^* is the solution of (4.1). It is not difficult to verify that this solution is unique. We have solved system (4.1) by the gradient-based (GB) [1], cyclic [19] and cyclic oblique projection (Cylic-OP) methods. Numerical results for different values of n ($n = 100, 200, 300$ and 400) with $\delta = 10^{-7}$ are given in Table 1. In this table, we report the number of iterations (Iters) for the convergence, CPU times (in seconds) for the convergence and the parameter μ_{exp} where the experimentally found optimal parameters μ_{exp} are the ones resulting in the least numbers of iterations for the gradient-based and the cyclic methods. Moreover, we also give $\|\tilde{X}_1 - X_1^*\|_\infty$ in Table 1 where $\tilde{X}_1$ is the estimated solution by each of the methods. As seen the cyclic oblique projection method in terms of both number of iterations and CPU times is superior to the other two methods. The convergence history of the three methods for $n = 400$ are depicted in Figure 1. In this figure, $\log_{10} \|X_1(k) - X_1(k-1)\|_\infty$ is presented in terms of iterations.

Example 4.2. In the current instance, we focus on the coupled linear matrix equations

$$\begin{cases} A_{11}X_1B_{11} + C_{11}X_1^T D_{11} = F_1, \\ A_{21}X_1B_{21} + C_{21}X_1^T D_{21} = F_2, \end{cases} \tag{4.2}$$

Table 1. Numerical results for Example 4.1.

	n	100	200	300	400
GB	Iters	499	514	516	514
	CPU time	2.68	17.39	63.45	138.53
	$\|\tilde{X}_1 - X_1^*\|_\infty$	2.75e-6	2.79e-6	2.88e-6	3.09e-6
	$\mu_{\exp}$	0.0159	0.0159	0.0159	0.0159
Cyclic	Iters	953	973	975	975
	CPU time	4.05	25.41	89.91	202.33
	$\|\tilde{X}_1 - X_1^*\|_\infty$	2.93e-6	2.28e-6	3.45e-6	3.45e-6
	$\mu_{\exp}$	0.0164	0.0164	0.0164	0.0164
Cyclic-OP	Iters	187	215	225	215
	CPU time	1.14	8.20	30.59	65.58
	$\|\tilde{X}_1 - X_1^*\|_\infty$	2.00e-7	1.04e-7	9.37e-8	6.22e-8

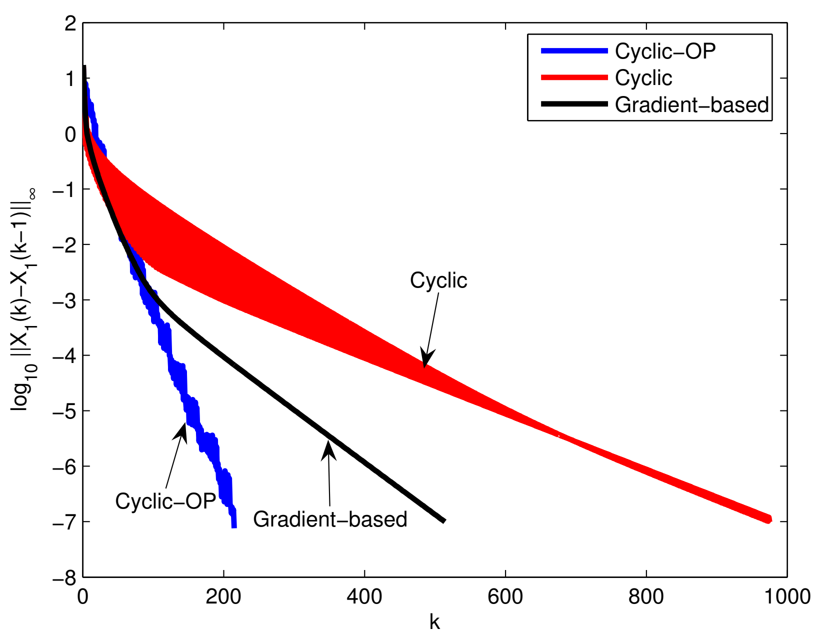

Figure 1. $\log_{10}\|X_1(k) - X_1(k-1)\|_\infty$ for Example 4.1 with $n = 400$).

where

$$A_{11} = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}, \quad B_{11} = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad C_{11} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix},$$

$$D_{11} = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 3 & 1 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 2 & 3 & -4 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$C_{21} = \begin{pmatrix} -1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \qquad D_{21} = \begin{pmatrix} 3 & 3 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix},$$

and

$$F_1 = \frac{1}{9}\begin{pmatrix} 48 & -8 & 24 \\ 60 & 20 & 44 \\ 132 & 92 & 100 \end{pmatrix}, \quad F_2 = \frac{1}{9}\begin{pmatrix} 112 & 80 & 400 \\ 40 & 92 & 40 \\ 124 & 116 & 340 \end{pmatrix}.$$

Suppose that the reflection matrices P_1 and Q_1 are given as follows:

$$P_1 = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix} \quad \text{and} \quad Q_1 = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}.$$

It can be checked that system (4.2) has infinitely number of solutions which are centro-symmetric with respect to P_1 and Q_1. Two of these solutions are given as follows

$$\tilde{X}_1^* = \frac{1}{9}\begin{pmatrix} -4 & 16 & 4 \\ -4 & 7 & 13 \\ -16 & 1 & 7 \end{pmatrix} \quad \text{and} \quad \bar{X}_1^* = \frac{1}{9}\begin{pmatrix} -4 & 16 & 4 \\ -4 & 16 & 4 \\ -16 & -8 & 16 \end{pmatrix}.$$

In this example, the tolerance δ is set to be 10^{-12}. We choose two different initial guesses and present the corresponding results. We first use a zero matrix as the initial guess. All of the other assumptions are as the previous example. In this case, all the three methods converge to the solution $\tilde{X}_1^*$. As the previous example the computed solution by the methods is denoted by $\tilde{X}_1$. The numerical results are reported in Table 2. The numerical experiments of the cyclic and gradient-based methods demonstrate that they can not compete with the proposed method. For more elucidation, the convergence history of the methods are illustrated in Figure 2.

We now consider the matrix $X_1(0) = I + P_1Q_1$ as an initial guess, where I is the identity matrix. It is noted that the matrix $X_1(0)$ is centro-symmetric with respect to P_1 and Q_1. In this case all of the three methods converge to the solution $\bar{X}_1^*$. Numerical results are given in Table 3. As observed the cyclic oblique projection method is superior to the other two methods in terms of both iterations and CPU times. For more clarification, we exhibit the convergence curves of the methods in Figure 3.

Remark 4.3. *In the reported example in [19], it can be observed that the proposed cyclic method outperformed the gradient-based method for solving the mentioned coupled linear matrix equation. Nevertheless, we have numerically collated the performance of these algorithms for several examples. As it can be also seen in our presented examples, although the cyclic method has wider convergence region than the gradient-based algorithm, in most of our examined examples the gradient-based method surpasses the cyclic method when the optimum values of the fixed parameters of the algorithms are utilized. However in all of our executed numeric experiments the presented cyclic method with oblique projection technique (Cyclic-OP) outperforms the gradient-based and cyclic methods. We would like to comment here that the optimum values of the fixed parameters have been experimentally selected for the gradient-based and cyclic methods to solve the considered coupled Sylvester-transpose matrix equations over centro-symmetric matrices. The open problem of determining the optimum value of the fixed parameters of the algorithms in these situations may be a subject of interest. However, we have illustrated the superior convergence behavior of the Cyclic-OP method in comparison with the gradient-based and cyclic methods with their best convergence factors.*

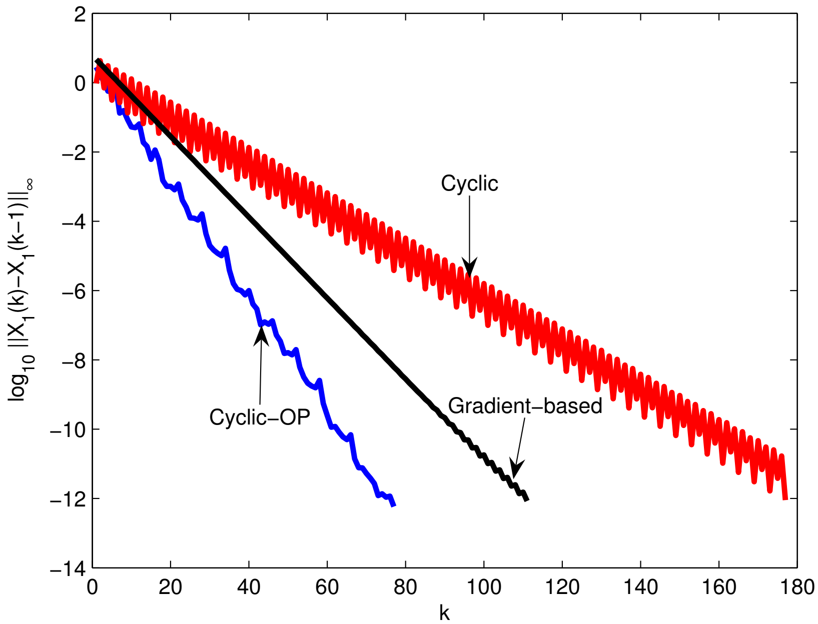

FIGURE 2. $\log_{10} \|X_1(k) - X_1(k-1)\|_\infty$ for Example 4.2 for zero initial guess.

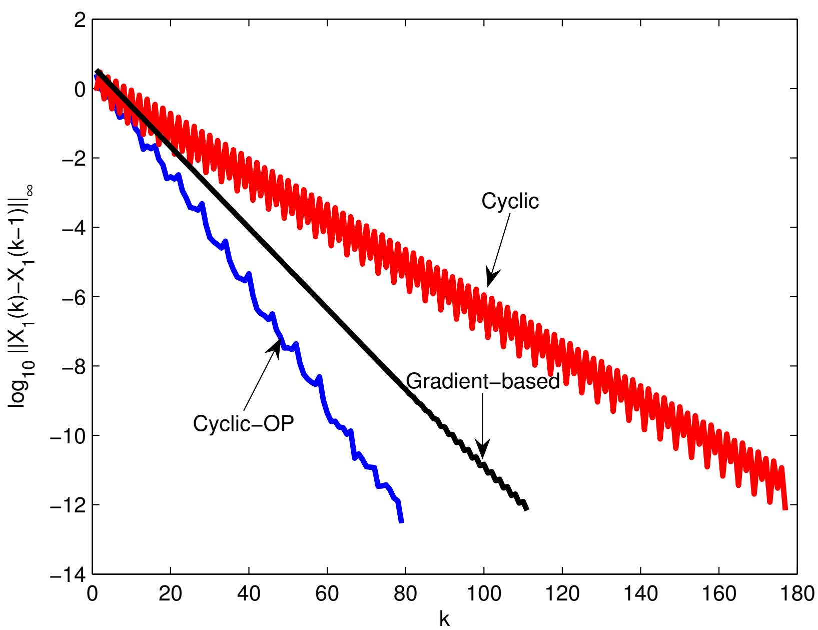

FIGURE 3. $\log_{10} \|X_1(k) - X_1(k-1)\|_\infty$ for Example 4.2 for initial guess $X_1(0) = I + P_1Q_1$.

TABLE 2. Numerical results for Example 4.2 with zero initial guess.

	GB	Cyclic	Cyclic-OP
Iters	111	177	77
CPU time	0.02	0.02	0.02
$\|\tilde{X}_1 - \tilde{X}_1^*\|_\infty$	4.46e-12	2.66e-11	1.11e-12
$\mu_{\exp}$	0.0029	0.00336	–

TABLE 3. Numerical results for Example 4.2 with initial guess $X_1(0) = I + P_1Q_1$.

	GB	Cyclic	Cyclic-OP
Iters	111	177	79
CPU time	0.02	0.02	0.02
$\|\tilde{X}_1 - \bar{X}_1^*\|_\infty$	3.71e-12	1.33e-11	7.40e-13
$\mu_{\exp}$	0.0029	0.00336	–

5. Conclusion

We have firstly developed the cyclic iterative method to determine the unique centro-symmetric solution group of the coupled Sylvester-transpose matrix equations and analyzed the convergence properties of the proposed algorithm. Afterwards, the assumption of the existence of the unique solution has been discarded. Meanwhile, an oblique projection technique has been exploited to present a new modified cyclic iterative method. It has been both theoretically and experimentally illustrated that our offered approach can ameliorate the speed of the convergence of the cyclic iterative method which incorporates the algorithm proposed by Tang et al. [Numer. Algorithms, 66 (2014), No. 2, 379–397] whereas we have not set the restriction of the existence of unique solution.

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References

[1] F. P. A. Beik, D. K. Salkuyeh and M. M. Moghadam, Gradient-based iterative algorithm for solving the generalized coupled Sylvester-transpose and conjugate matrix equations over reflexive (anti-reflexive) matrices, *Transactions of the Institute of Measurement and Control,* 36 (2014), No. 1, 99–110.

[2] D. P. Bertsekas, A new class of incremental gradient methods for least squares problems, *SIAM J. Optimiz*, 7 (1997), No. 4, 913–926.

[3] M. Dehghan and M. Hajarian, Solving the generalized Sylvester matrix equation $\sum_{i=1}^{p} A_iXB_i + \sum_{j=1}^{q} C_jYD_j = F$ over reflexive and anti-reflexive matrices, *International Journal of Control, Automation, and Systems*, 9 (2011), No. 1, 118–124.

[4] F. Ding and T. Chen, Hierarchical identification of lifted state-space models for general dual-rate systems, *IEEE Transactions on Circuits and Systems–I: Regular Papers,* 52 (2005), No. 6, 1179–1187.

[5] F. Ding and T. Chen, Hierarchical gradient-based identification of multivariable discrete-time systems, *Automatica,* 41 (2005), No. 2, 315–325.

[6] F. Ding and T. Chen, Hierarchical least squares identification methods for multivariable systems, *IEEE Transactions on Automatic Control,* 50 (2005), No. 3, 397–402.

[7] F. Ding and T. Chen, Gradient based iterative algorithms for solving a class of matrix equations, *IEEE Transactions on Automatic Control,* 50 (2005), No. 8, 1216–1221.

[8] F. Ding and T. Chen, Iterative least squares solutions of coupled Sylvester matrix equations, *Syst. Contr. Lett.* 54 (2005), No. 2, 95–107.
[9] F. Ding and T. Chen, On iterative solutions of general coupled matrix equations, *SIAM J. Control Optim.* 44 (2006), No. 6, 2269–2284.
[10] F. Ding, P.X. Liu and J. Ding, Iterative solutions of the generalized Sylvester matrix equations by using the hierarchical identification principle, *Appl. Math. Comput.* 197 (2008), No. 1, 41–50.
[11] J. Ding, Y. J. Liu and F. Ding, Iterative solutions to matrix equations of the form $A_iXB_i = F_i$, *Comput. Math. Appl.* 59 (2010), No. 11, 3500–3507.
[12] F. Ding, Y. Liu, and B. Bao, Gradient-based and least-squares-based iterative estimation algorithms for multi-input multi-output systems, *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering,* 226 (2013), No. 1, 43–55.
[13] F. Ding, X. Liu, J. Chu, Gradient-based and least-squares-based iterative algorithms for Hammerstein systems using the hierarchical identification principle, *IET Control Theory* & *Applications,* 7 (2013), No. 2, 176–184.
[14] Z. Y. Li and Y. Wang, Iterative algorithm for minimal norm least squares solution to general linear matrix equations, *Int. J. Comput. Math.* 87 (2010), No. 11, 2552–2567.
[15] A. Nedić and D. P. Bertsekas, Incremental subgradient methods for nondifferentiable optimization, *SIAM J. Optimiz*, 12 (2001), No.1, 109–138.
[16] D. K. Salkuyeh and F. P. A. Beik, On the gradient based algorithm for solving the general coupled matrix equations, *Transactions of the Institute of Measurement and Control,* 36 (2014), No. 3, 375–381.
[17] D. K. Salkuyeh and F. P. A. Beik, Minimum norm least-squares solution to general complex coupled linear matrix equations via iteration, *Filomat,* 29 (2015), No. 6, 1389–1407.
[18] C. Song, G. Chen and L. Zhao, Iterative solutions to coupled Sylvester-transpose matrix equations, *Appl. Math. Model.* 35 (2011), No. 10, 4675–4683.
[19] Y. Tang, J. Peng and S. Yue, Cyclic and simultaneous iterative methods to matrix equations of the form $A_iXB_i = F_i$, *Numer. Algorithms,* 66 (2014), No. 2, 379–397.
[20] W. Xiong, J. Ma and R. Ding, An iterative numerical algorithm for modeling a class of Wiener nonlinear systems, *Appl. Math. Lett.* 26 (2013), No. 4, 487–493.
[21] L. Zhou, X. Li and F. Pan, Gradient-based iterative identification for MISO Wiener nonlinear systems: Application to a glutamate fermentation process, *Appl. Math. Lett.* 26 (2013), No. 8, 886-892.