The generalized Hermitian and skew-Hermitian splitting iterative method for image restoration

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Abstract

In this paper, we apply the generalized Hermitian and skew-Hermitian splitting (GHSS) iterative method to the problem of image restoration. We use a new splitting of the Hermitian part of the coefficient matrix of the problem. Moreover, we introduce a restricted version of the GHSS (RGHSS) iterative method together with its convergence properties. The optimal parameter, which minimizes the spectral radius of the iteration matrix of the RGHSS method, is also given. We present some numerical examples to show the effectiveness and accuracy of the method and compare it with a recently proposed method.

Key words: Image restoration, boundary conditions, Hermitian and skew-Hermitian splitting method, generalized HSS, restricted GHSS, convergence


1. Introduction

The history of image restoration dates back to 1950s and early 1960s when scientists from the United States and former Soviet Union who were involved in space programs primarily introduced the field [1]. Since then, the applications of this field have been further investigated in several fields of applied sciences such as: removing the noise in magnetic resonance imaging (MRI), chest X-rays and digital angiographic images in medical imaging [2, 3], restoration of aging and deteriorated films in engineering [4], restoring degraded images obtained by telescopes or satellites in astronomy [5], restoration of degraded images in optical systems in optics [6], and many other areas (see [7, 8]).

Restoration is the process of reconstructing or recovering a degraded image by using a priori knowledge of the degradation phenomenon. An input-output relationship for image restoration can be written as follows [9]:

\[ g(x, y) = \mathcal{H}[f(x, y)] + n(x, y), \]  

(1)

where \( \mathcal{H} \) is a degradation operator, \( f(x, y) \) is the original image, \( g(x, y) \) is the degraded image (recorded image) and \( n(x, y) \) is additive noise. It can be shown that if \( \mathcal{H} \) is a linear
and space-invariant operator, then Eq. (1) can be written as the Fredholm integral equation of first kind

\[ g(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(x - \xi, y - \eta) f(\xi, \eta) d\xi d\eta + n(x, y), \]

where \( h(x, y) \) is commonly called the point spread function (PSF) and \( n(x, y) \) is independent of spatial coordinates. In some cases, a precise mathematical expression exists for the PSF arrays. A case is the PSF for the degraded image by atmospheric turbulence, which is considered a two-dimensional Gaussian function [10]. The Moffat function is used for modeling the PSF of an astronomical telescope [11]. In some other cases, the PSF arrays can be estimated from degraded image [12, 13]. In this paper, the first case is considered in the image restoration process. By discretizing the functions and approximating integration with a quadrature rule, Eq. (2) can be written in the following form:

\[ g(x, y) = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} h(x - k, y - l) f(k, l) + n(x, y). \]

In order to make Eq. (3) convenient so that we can apply numerical linear algebra or statistics methods, the matrix-vector form of the equation is written as:

\[ g = Af + \eta, \]

where \( A \) is a blurring matrix of size \( n^2 \times n^2 \) and \( f, \eta, \) and \( g \) are \( n^2 \)-dimensional vectors representing the original image, noise, and blurred and noisy image, respectively. It is noted that the observed image \( g \) is of finite dimension in practice. Hence, a finite section of original image \( f \) is considered for image restoration. Therefore, since Eq. (3) is a convolution, some assumptions are considered on the function \( f \) outside the observed image domain called as boundary conditions (BCs). These assumptions lead to some structures for the matrix \( A \). There are various choices for the matrix \( A \) depending on how we consider the BCs. In this work, the following BCs are considered [8, 14, 15, 16, 17, 18]:

- **Zero BC**: The simplest case to choose the BC is zero one. For this case, the outside boundary of the exact image is considered black (i.e. it consists of zeros). It can be seen that the blurring matrix \( A \) is a block Toeplitz with Toeplitz blocks (BTTB) matrix. Fast Fourier Transforms (FFTs) are effectively applied to matrix-vector multiplications in zero BC.

- **Periodic BC**: The data outside the domain of consideration are periodically extended. In this case, the structure of matrix \( A \) is block circulant with circulant blocks (BCCB). Similar to zero BC, FFTs are effective methods for matrix-vector multiplications involving BCCB matrices.

- **Reflexive BC**: In this case, it is assumed that the original scene inside the boundary immediately reflects itself outside of the boundary. Matrix \( A \) has a block Toeplitz-plus-Hankel with Toeplitz-plus-Hankel blocks (BTHTHB) structure. The two-dimensional fast cosine transform (DCT-III) can be used to diagonalize the matrix \( A \) when the PSF is symmetric.
• Antireflexive BC: For the antireflexive BC, the data outside domain of consideration are antireflection data inside the viewable region, i.e. 
\[ f_{1-j} - f_1 = -(f_{j+1} - f_1), \quad f_{n^2+j} - f_{n^2} = -(f_{n^2-j} - f_{n^2}), \quad j = 1, \ldots, n^2. \]
In this case, it can be seen that the coefficient matrix \( A \) is a block Toeplitz-plus-Hankel-plus-rank-2-correction, with Toeplitz-plus-Hankel-plus-rank-2-correction blocks. If the PSF is symmetric, the discrete sine transform (DST-I) can be used to diagonalize the coefficient matrix \( A \).

• Mean BC: This BC can be considered as an adaptive antireflection which preserves \( C^1 \) continuity and reduces the ringing effects. Matrix \( A \) has a block-Toeplitz-plus-Rank4 with Toeplitz-plus-Rank4-blocks (BTR4TR4B) structure.

Due to the ill-posedness of model (1), the linear system (4) is ill-conditioned. One of the most effective methods for solving ill-conditioned linear systems is the Tikhonov method [19]. The Tikhonov regularized solution is given by solving the following problem:

\[
\min_f \| Af - g \|_2^2 + \mu^2 \| Lf \|_2^2,
\]
(5)

where \( L \) is a carefully chosen regularization matrix and \( \mu > 0 \) is called the regularization parameter. In the Tikhonov method, the factor \( \mu \) controls the balance between the minimization of \( \| Af - g \|_2^2 \) and the regularization term \( \| Lf \|_2^2 \), involves a smoothing norm. Note that the regularization parameter is generally small and calculated by noisy level. In this work, we consider Eq. (5) when \( 0 < \mu < 1 \) and \( L = I \). It can be easily seen that the Tikhonov minimization problem is mathematically equivalent to solving the following equation [8]:

\[
(A^T A + \mu^2 I)f = A^T g.
\]
(6)

Several iterative methods have been presented to investigate the solution of Eq. (6) in the literature. In [20], Lv et al. presented the following equivalent system

\[
\begin{bmatrix}
I & A \\
-A^T & \mu^2 I
\end{bmatrix}
\begin{bmatrix}
e \\
f
\end{bmatrix}
=
\begin{bmatrix}
g \\
0
\end{bmatrix},
\]
(7)

where \( K \) is \( 2n^2 \times 2n^2 \) non-Hermitian positive definite matrix and the auxiliary variable, \( e \), represents the additive noise, i.e., \( e = g - Af \). They used the idea of the Hermitian and skew-Hermitian (HSS) iterative method and presented a special case of the HSS (SHSS) method to solve the proposed equation.

In [21], Bai et al. presented the HSS iterative method for non-Hermitian positive definite linear systems. The preconditioned version of the HSS method and the convergence results have been presented in [22, 23]. Moreover, some modified forms of the HSS method are given in [24, 25]. In 2009, Benzi presented a generalization of the HSS (GHSS) iterative method to solve a class of non-Hermitian linear systems [26]. The GHSS method is based on the splitting of the Hermitian part of the coefficient matrix of linear system to positive definite and positive semi-definite matrices. A generalization of the HSS method for solving singular linear systems is also given in [27].

In this paper, we propose a restricted version of the GHSS (RGHSS) method to solve Eq. (7). Convergence of the method along with the application of the method to the problem of image restoration are investigated.
This paper is organized as follows. In Section 2, we give a brief description of the HSS, SHSS and GHSS methods to solve linear systems. A new splitting of iteration matrix for the application of GHSS method is presented in Section 3. Some convergence results are also considered in this section. To show the effectiveness and accuracy of the proposed method, we present three numerical tests in Section 4. Finally, some concluding remarks are presented in Section 5.

2. A brief description of HSS, SHSS and GHSS

Let $A$ be a non-Hermitian matrix. In the HSS method, first the matrix $A$ is split as

$$A = H + S,$$  \hspace{1cm} (8)

where

$$H = \frac{1}{2}(A + A^T), \quad S = \frac{1}{2}(A - A^T).$$  \hspace{1cm} (9)

It is noted that since the coefficient matrix of the problem under investigation is real, hereafter, we use the subscript “T” instead of “H”. Then, the two splittings of $A$ are presented as:

$$A = (H + \alpha I) - (\alpha I - S), \quad A = (S + \alpha I) - (\alpha I - H),$$ \hspace{1cm} (10)

where $\alpha > 0$. Finally, the HSS iteration is given by alternating between the two splittings for solving the proposed system as

$$\begin{cases}
(H + \alpha I)x_{k+\frac{1}{2}} = (\alpha I - S)x_k + b, \\
(S + \alpha I)x_{k+1} = (\alpha I - H)x_{k+\frac{1}{2}} + b,
\end{cases} \quad k = 0, 1, \ldots,$$ \hspace{1cm} (11)

where $x_0$ is a given initial guess. The SHSS iterative method is given by substituting $\alpha := 1$ in the second equation of Eq. (11). Lv et al in [20] showed that the SHSS method is an effective and accurate method for image restoration.

In the GHSS method the Hermitian part of $H$ is decomposed as

$$H = G + P = \epsilon L + P,$$ \hspace{1cm} (12)

where $L$ is a Hermitian positive definite, $P$ is a Hermitian positive semidefinite and $\epsilon > 0$ is a small constant. By choosing an initial guess $x_0$, the GHSS iteration is given as follows:

$$\begin{cases}
(G + \alpha I)x_{k+\frac{1}{2}} = (\alpha I - S)x_k + b, \\
(S + P + \alpha I)x_{k+1} = (\alpha I - G)x_{k+\frac{1}{2}} + b.
\end{cases} \quad k = 0, 1, \ldots,$$ \hspace{1cm} (13)

In the following section, the GHSS iterative method along with a restricted version of it are applied to solve Eq. (7).
3. Application of GHSS in image restoration

For the image restoration problem presented in Eq. (7), we have

\[ K = \begin{bmatrix} I & A \\ -A^T & \mu^2I \end{bmatrix} = \begin{bmatrix} I & O \\ O & \mu^2I \end{bmatrix} + \begin{bmatrix} O & A \\ -A^T & O \end{bmatrix} = H + S. \]  

(14)

Now, we apply the GHSS method to the problem (7). To this end, we present a splitting for the Hermitian part of \( K \) based on GHSS idea. The splitting used for the matrix \( K \) in the GHSS method is as follows:

\[ K = \beta \begin{bmatrix} I & O \\ O & \mu^2I \end{bmatrix} + \begin{bmatrix} (1 - \beta)I & O \\ O & (1 - \beta)\mu^2I \end{bmatrix} + \begin{bmatrix} O & A \\ -A^T & O \end{bmatrix} = G + P + S, \]  

(15)

where \( \beta \) is a positive constant. We use this splitting to implement the GHSS iteration (13). Therefore, the solution of the first equation in (13) can be obtained by matrix-vector multiplication as:

\[ x_{k+\frac{1}{2}} = \begin{bmatrix} \frac{1}{\beta + \alpha}I & O \\ O & \frac{1}{\beta \mu^2 + \alpha}I \end{bmatrix} \begin{bmatrix} (\alpha + \beta - 1)I & -A \\ A^T & \alpha + (\beta - 1)\mu^2I \end{bmatrix} x_k + b. \]  

(16)

It should be noted that the Krylov subspace methods are effective on problems containing matrix-vector multiplications. On the other hand, the second equation has a special structure such as block Toeplitz or block circulant. Hence, the GMRES method can be used to solve the second equation with acceptable operation complexity [28]. Finally, the GHSS algorithm can be written as follows through some simple manipulations of proposed algorithm in [20].

Algorithm 1: The GHSS iterative method

1. Choose the initial guess of original image \( f_0 = g \), initial value of noise \( e_0 = g - Af_0 \), maximum number of outer iteration \( M \) and very small positive \( \tau \).
2. \( r_0 := b - Kx_0 \);
3. For \( k = 0, 1, 2, \ldots \), until \( \frac{\|r_k\|_2}{\|r_0\|_2} > \tau \) or \( k < M \) Do,
   \[ e_{k+\frac{1}{2}} := \frac{1}{\beta + \alpha}((\alpha + \beta - 1)e_k - Af_k + g), \]
   \[ f_{k+\frac{1}{2}} := \frac{1}{\beta \mu^2 + \alpha}(A^T e_k + (\alpha + (\beta - 1)\mu^2)f_k), \]
   \[ \text{Solve } \begin{cases} (\alpha + 1 - \beta)e_{k+1} + Af_{k+1} = (\alpha - \beta)e_{k+\frac{1}{2}} + g, \\
   -A^T e_{k+1} + (\alpha + (1 - \beta)\mu^2)f_{k+1} = (\alpha - \beta\mu^2)f_{k+\frac{1}{2}}. \end{cases} \]
   \[ r_{k+1} := b - Kx_{k+1}, \]
4. EndDo

Remark 1: In the third step of Algorithm 1, we solve the proposed system by some Krylov subspace methods such as the well-known GMRES method. To this end, we use the initial
guesses \( e_{k+1}^{(0)} = e_{k+1/2} \) and \( f_{k+1}^{(0)} = f_{k+1/2} \). Moreover, the applied iterated method acts until \( \| q_j \|_2 / \| q_0 \|_2 < \zeta \), where \( \zeta \) is a very small positive value and the residual \( q_j \) is defined as:

\[
q_j = \begin{bmatrix}
(\alpha - \beta)e_{k+\frac{1}{2}} + g - (\alpha + 1 - \beta)e_{k+1} - Af_{k+1}^{(j)} \\
(\alpha - \beta \mu^2)f_{k+\frac{1}{2}} + A^T e_{k+1}^{(j)} - (\alpha + (1 - \beta) \mu^2)f_{k+1}^{(j)}
\end{bmatrix}.
\]

Now, we present the following theorem for the convergence of the proposed GHSS iterative method to solve Eq. (7).

**Theorem 1.** Let \( 0 < \beta \leq 1 \). Assume that the matrices \( G, P \) and \( S \) are defined as Eq. (15). Then the iteration (13) unconditionally converges to the unique solution of \( Kx = b \).

**Proof.** For \( 0 < \mu, \beta \leq 1 \), \( G \) is Hermitian positive definite, \( P \) is positive semidefinite and \( S \) is skew-Hermitian part of splitting. Then, the proposed method unconditionally converges to the unique solution of \( Kx = b \) according to Theorem 2.2 in [26]. \( \square \)

### 3.1. Restricted version of the GHSS method

Similar to the idea proposed by Lv et al. [20], we present the RGHSS method. Since, finding the optimal values for \( \alpha \) and \( \beta \) is generally very complicated, we fixed the parameter \( \alpha \) in the second equation of the GHSS method (13) at \( \alpha = \beta \) as

\[
\begin{align*}
(G + \alpha I)x_{k+\frac{1}{2}} &= (\alpha I - P - S)x_k + b, \\
(S + P + \beta I)x_{k+1} &= (\beta I - G)x_{k+\frac{1}{2}} + b,
\end{align*}
\]

where \( \beta \) is a given positive number. Note that by fixing \( \alpha \) in the second equation, we can set the \( n^2 \) eigenvalues of the iteration matrix equal to zero and control \( n^2 \) others. To investigate the convergence of the method we first recall the next lemma.

**Lemma 2.** Let \( x_0 \) be a given initial vector, \( A \in \mathbb{R}^{n \times n} \), and two splittings of the matrix \( A \) are given as \( A_i = M_i - N_i \) (\( i=1,2 \)). If \( x_k \) is a two-step iteration sequence defined by

\[
\begin{align*}
M_1x_{k+\frac{1}{2}} &= N_1x_k + b, \\
M_2x_{k+1} &= N_2x_{k+\frac{1}{2}} + b,
\end{align*}
\]

\( k = 0, 1, \ldots \), then

\[
x_{k+1} = M_2^{-1}N_2M_1^{-1}N_1x_k + M_2^{-1}(I + N_2M_1^{-1})b, \quad k = 0, 1, \ldots
\]

Moreover, if the spectral radius \( \rho(M_2^{-1}N_2M_1^{-1}N_1) < 1 \), then the iterative sequence \( \{x_k\} \) converges to the unique solution \( x^* \in \mathbb{R}^n \) of \( Ax = b \) for all initial vectors \( x_0 \in \mathbb{R}^n \).

Note that unlike the previous discussion, the constant \( \beta \) is not bounded and it is considered as a positive real constant for the RGHSS method. Hence, the existence of the following theorem is necessary in this study.
Theorem 3. Let $K \in \mathbb{R}^{2n^2 \times 2n^2}$. Assume that the matrices $G$, $P$ and $S$ are defined as Eq. (15). Let also $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n^2}$ be the singular values of $A$. Then, the iteration matrix of RGHSS method is given by

$$T(\alpha, \beta) = (\beta I + P + S)^{-1}(\beta I - G)(\alpha I + G)^{-1}(\alpha I - P - S).$$

(20)

Furthermore, $T(\alpha, \beta)$ has the following properties:

i) $T(\alpha, \beta)$ has a zero eigenvalue with multiplicity $n^2$.

ii) The other $n^2$ eigenvalues of $T(\alpha, \beta)$ are given by

$$\eta_i = \frac{\beta(1 - \mu^2)\alpha - \mu^2(1 - \beta) - \sigma_i^2}{\alpha + \beta\mu^2\beta + \mu^2(1 - \beta) + \sigma_i^2}, \quad i = 1, \ldots, n^2.$$  

(21)

iii) If $0 < \beta \leq 1 + (\sigma_{n^2}/\mu)^2$, then for

$$\alpha > \frac{(\sigma_1^2 + \mu^2}\beta - 2\beta\mu^2(\sigma_1^2 + \beta + \mu^2(1 - \beta))}{\sigma_1^2 + \mu^2 + 2\beta(1 - \mu^2)} = \alpha_1.$$  

(22)

we have $\rho(T(\alpha, \beta)) < 1$. Hence, the RGHSS iteration method converges for any initial vector $x_0$. In this case, the optimal parameter $\alpha$ is given by

$$\alpha^* = \arg \min_{\alpha} \rho(T(\alpha, \beta)) = \frac{\beta(1 - \mu^2)}{\beta + (1 - \beta)\mu^2 + \sigma_{n^2}^2} \arg \min_{\alpha} \left(1 - \frac{\mu^2 + \sigma_{n^2}^2}{\alpha + \beta\mu^2}\right).$$

Therefore, as $\alpha \to 0$, we have

$$\alpha^* \to \frac{\beta(1 - \mu^2)}{\beta + (1 - \beta)\mu^2 + \sigma_{n^2}^2} \left(1 - \frac{\mu^2 + \sigma_{n^2}^2}{\beta\mu^2}\right).$$

iv) If $\beta \geq 1 + (\sigma_1/\mu)^2$, then for every positive number $\alpha$, the RGHSS iterative method is convergent and we have

$$\alpha^* = \arg \min_{\alpha} \rho(T(\alpha, \beta)) = \frac{\beta(1 - \mu^2)}{\beta + (1 - \beta)\mu^2 + \sigma_{n^2}^2} \arg \min_{\alpha} \left(1 - \frac{\mu^2 + \sigma_{n^2}^2}{\alpha + \beta\mu^2}\right).$$

v) If $1 + (\sigma_{n^2}/\mu)^2 < \beta < 1 + (\sigma_1/\mu)^2$, then for $\alpha > \alpha_1$ the RGHSS iterative method is convergent.

Proof. It can be easily seen that $\alpha I + G$ and $\beta I + P + S$ are nonsingular for any positive constant numbers $\alpha$ and $\beta$. By putting

$$M_1 = \alpha I + G, \quad N_1 = \alpha I - P - S, \quad M_2 = \beta I + P + S, \quad N_2 = \beta I - G,$$

in Lemma 2, the proposed $T(\alpha, \beta)$ is obtained. Now, let

$$\hat{T}(\alpha, \beta) = (\beta I - G)(\alpha I + G)^{-1}(\alpha I - P - S)(\beta I + P + S)^{-1}.$$  

(23)
Since the eigenvalue distribution of $T(\alpha, \beta)$ and $\hat{T}(\alpha, \beta)$ are the same, it is sufficient to discuss the eigenvalues of $\hat{T}(\alpha, \beta)$ in the sequel. Considering the definition of $G$, $P$, and $S$ from Eq. (15), we have

$$(\alpha I + G)^{-1} = \begin{bmatrix} \frac{1}{2} I & O \\ \frac{1}{\alpha + \beta I^2} I \end{bmatrix}, \quad \alpha I - P - S = \begin{bmatrix} (\alpha + \beta - 1)I & -A \\ AT & (\alpha + \mu^2(\beta - 1))I \end{bmatrix},
$$

$$(\beta I + P + S)^{-1} = \begin{bmatrix} I - AZ^{-1}AT & -AZ^{-1} \\ Z^{-1}AT & Z^{-1} \end{bmatrix}, \quad \beta I - G = \begin{bmatrix} O & O \\ O & \beta(1 - \mu^2)I \end{bmatrix},$$

where $Z = (\beta(1 - \mu^2) + \mu^2)I + AT A$. Substituting these matrices into Eq. (23), and after some straightforward computations, we have

$$\hat{T}(\alpha, \beta) = \begin{bmatrix} O & O \\ \Lambda & \Theta \end{bmatrix},$$

where

$$\Theta = \frac{\beta(1 - \mu^2)}{\alpha + \beta \mu^2} \left( (\alpha - \mu^2(1 - \beta))I - AT A \right) \left( (\beta + \mu^2(1 - \beta))I + AT A \right)^{-1},$$

and $\Lambda$ is an $n^2 \times n^2$ matrix which is not the focus of our discussion. As we see, zero is an eigenvalue of $\hat{T}(\alpha, \beta)$ with multiplicity $n^2$. The other $n^2$ eigenvalues of $\hat{T}(\alpha, \beta)$ are equal to those of $\Theta$. It can be shown that the eigenvalues of $\Theta$ are given by

$$\eta_i = \frac{\beta(1 - \mu^2)}{\alpha + \beta \mu^2} \frac{\alpha - \mu^2(1 - \beta) - \sigma_i^2}{\alpha + \beta \mu^2(1 - \beta) + \sigma_i^2}, \quad i = 1, \ldots, n^2,$$

where $\sigma_i$'s are the singular values of the matrix $A$. To prove the convergence, we have to show that $\rho(\hat{T}(\alpha, \beta)) < 1$ under the condition provided by the theorem. We have

$$\rho(\hat{T}(\alpha, \beta)) = \rho((\beta I - G)(\alpha I + G)^{-1}(\alpha I - P - S)(\beta I + P + S)^{-1})$$

$$= \max_{\sigma_i \in \sigma(A)} \frac{\beta(1 - \mu^2)}{\alpha + \beta \mu^2} \frac{\alpha - \mu^2(1 - \beta) + \sigma_i^2}{\beta + \mu^2(1 - \beta) + \sigma_i^2}.$$

(24)

Now, if $0 < \beta \leq 1 + (\sigma_{n^2}/\mu)^2$, then $\mu^2(1 - \beta) + \sigma_i^2 \geq 0, i = 1, \ldots, n^2$. Therefore, there exists $\hat{\alpha} > 0$ such that

$$\max_{\sigma_i \in \sigma(A)} \frac{\alpha - \mu^2(1 - \beta) + \sigma_i^2}{\beta + \mu^2(1 - \beta) + \sigma_i^2} = \begin{cases} \frac{\alpha - (\mu^2(1 - \beta) + \sigma_i^2)}{\beta + \mu^2(1 - \beta) + \sigma_i^2} & \alpha \geq \hat{\alpha}, \\ \frac{\sigma_i^2 + \mu^2(1 - \beta) - \alpha}{\beta + \mu^2(1 - \beta) + \sigma_i^2} & \alpha \leq \hat{\alpha}. \end{cases}$$

(25)

If $\alpha \geq \hat{\alpha}$, it can be shown that for $\alpha > -\beta$, $\rho(\hat{T}(\alpha, \beta)) < 1$. Otherwise, if $\alpha \leq \hat{\alpha}$, by setting $\frac{\sigma_i^2 + \mu^2(1 - \beta) - \alpha}{\beta + \mu^2(1 - \beta) + \sigma_i^2} < 1$, it can be easily seen that $\rho(\hat{T}(\alpha, \beta)) < 1$ whenever

$$\hat{\alpha} \geq \alpha > \frac{(\sigma_1^2 + \mu^2)(\beta - 2\beta \mu^2(\mu_1^2 + \beta + \mu^2(1 - \beta))}{\sigma_1^2 + \mu^2 + 2\beta(1 - \mu^2)} = \alpha_1.$$

(26)
To find the optimal parameter $\alpha^*$ of the RGHSS method for any $0 < \beta \leq 1 + (\sigma_n^2/\mu)^2$, we minimize the spectral radius of iteration matrix. In other words, we have

$$\alpha^* = \arg \min_\alpha \rho(\hat{T}(\alpha, \beta)).$$  \hspace{1cm} (27)$$

From Eqs. (24) and (25), we see that

$$\rho(\hat{T}(\alpha, \beta)) = \max \left\{ \frac{\beta(1 - \mu^2)}{\alpha + \beta \mu^2} \frac{\sigma_1^2 + (\mu^2(1 - \beta) - \alpha)}{\beta + \mu^2(1 - \beta) + \sigma_1^2}, \frac{\alpha - (\mu^2(1 - \beta) + \sigma_n^2)}{\alpha + \beta \mu^2} \right\}.$$ 

For $\alpha^*$ to be the optimal point, it must be satisfied in the following equation

$$\frac{\sigma_1^2 + (\mu^2(1 - \beta) - \alpha^*)}{\beta + \mu^2(1 - \beta) + \sigma_1^2} = \frac{\alpha - (\mu^2(1 - \beta) + \sigma_n^2)}{\beta + \mu^2(1 - \beta) + \sigma_n^2}.$$ 

Hence, the optimal parameter $\alpha^*$ is given by

$$\alpha^* = \frac{2\mu^2(1 - \beta)(\beta + \mu^2(1 - \beta)) + (2\mu^2(1 - \beta) + \beta)(\sigma_1^2 + \sigma_n^2) + 2\sigma_1^2\sigma_n^2}{2\beta + 2\mu^2(1 - \beta) + \sigma_1^2 + \sigma_n^2}.$$ 

Now, if $\beta \geq 1 + (\sigma_1/\mu)^2$, then $-(\mu^2(1 - \beta) + \sigma_i^2) \geq 0$, $i = 1, \ldots, n^2$. In this case, it is easy to see that

$$\rho(\hat{T}(\alpha, \beta)) = \frac{\beta(1 - \mu^2)}{\alpha + \beta \mu^2} \frac{\max_{\sigma_i \in \sigma(A)} |\alpha - (\mu^2(1 - \beta) + \sigma_i^2)|}{\beta + \mu^2(1 - \beta) + \sigma_i^2} = \frac{\beta(1 - \mu^2)}{\alpha + \beta \mu^2} \frac{1}{\beta + \mu^2(1 - \beta) + \sigma_n^2}.$$ \hspace{1cm} (28)

It can be seen that $\alpha > -\beta$ is the necessary condition to have $\rho(\hat{T}(\alpha, \beta)) < 1$. Now, the optimal parameter $\alpha^*$ can be obtained as follows:

$$\alpha^* = \arg \min_\alpha \rho(\hat{T}(\alpha, \beta))$$

$$= \arg \min_\alpha \left\{ \frac{\beta(1 - \mu^2)}{\alpha + \beta \mu^2} \frac{\alpha - (\mu^2(1 - \beta) + \sigma_n^2)}{\beta + \mu^2(1 - \beta) + \sigma_n^2} \right\}$$

$$= \frac{\beta(1 - \mu^2)}{\beta + \mu^2(1 - \beta) + \sigma_n^2} \arg \min_\alpha \left\{ \frac{\alpha - (\mu^2(1 - \beta) + \sigma_n^2)}{\alpha + \beta \mu^2} \right\}$$

$$= \frac{\beta(1 - \mu^2)}{\beta + \mu^2(1 - \beta) + \sigma_n^2} \arg \min_\alpha \left\{ 1 - \frac{\mu^2 + \sigma_n^2}{\alpha + \beta \mu^2} \right\}.$$ 

In the last case, we consider $1 + (\sigma_n^2/\mu)^2 < \beta < 1 + (\sigma_1/\mu)^2$. Then

$$\sigma_i^2 - \sigma_1^2 < \mu^2(1 - \beta) + \sigma_i^2 < \sigma_n^2 \quad \text{for} \quad i = 1, \ldots, n^2.$$ 

Now suppose that for $i \geq k$, $\mu^2(1 - \beta) + \sigma_i^2 \geq 0$, then it can be shown that there exists an $\hat{\alpha} > 0$ such that

$$\max_{\sigma_i \in \sigma(A)} \frac{\alpha - (\mu^2(1 - \beta) + \sigma_i^2)}{\beta + \mu^2(1 - \beta) + \sigma_i^2} = \begin{cases} \frac{\alpha - (\mu^2(1 - \beta) + \sigma_i^2)}{\beta + \mu^2(1 - \beta) + \sigma_i^2} & \alpha \geq \hat{\alpha}, \\ \frac{\alpha^2 + \mu^2(1 - \beta) - \alpha}{\beta + \mu^2(1 - \beta) + \sigma_i^2} & \alpha \leq \hat{\alpha}. \end{cases}$$ \hspace{1cm} (29)
Also, for \( i \leq k \), we have
\[
\max_{\sigma_i \in \sigma(A)} \frac{|\alpha - (\mu^2(1 - \beta) + \sigma_i^2)|}{\beta + \mu^2(1 - \beta) + \sigma_i^2} = \frac{\alpha - (\mu^2(1 - \beta) + \sigma_i^2)}{\beta + \mu^2(1 - \beta) + \sigma_i^2}, \quad \alpha \leq \hat{\alpha}.
\] (30)

Similar to the previous cases, it is easy to see that for \( \alpha > \alpha_1 \), \( \rho(\hat{T}(\alpha, \beta)) < 1 \) and this completes the proof. \( \square \)

**Remark 2.** The SHHS method is a special case of the RGHHS method if we substitute \( \beta := 1 \). Therefore, the optimal value of \( \alpha \) for the SHSS method can be obtained from Theorem 3 which is
\[
\frac{\sigma_1^2 + \sigma_n^2 + 2\sigma_1^2\sigma_n^2}{2 + \sigma_1^2 + \sigma_n^2}.
\]

4. Illustrative examples

In this section, two new GHSS iterative methods are presented through two other splittings for Hermitian part of \( K \). Moreover, some examples are given to show the accuracy and effectiveness of the proposed methods. The proposed methods are implemented in MATLAB 7.0 software.

For further investigation, similar to GHSS method discussed in the previous section, two new splittings are considered for the Hermitian part of the matrix \( K \) as:

**GHSS-I**) In this case, the following splitting is considered:
\[
K = \mu^2 \begin{bmatrix} I & O & O \\ O & I & O \\ O & O & I \end{bmatrix} + \begin{bmatrix} (1 - \mu^2)I & O \\ O & I \\ O & O \end{bmatrix} + \begin{bmatrix} O & A \\ -A^T & O \end{bmatrix} = G + P + S.
\] (31)

**GHSS-II**) In the second case, the matrix \( K \) is split as follows:
\[
K = \beta \mu^2 \begin{bmatrix} I & O \\ O & I \end{bmatrix} + \begin{bmatrix} (1 - \beta \mu^2)I & O \\ O & (1 - \beta)\mu^2I \end{bmatrix} + \begin{bmatrix} O & A \\ -A^T & O \end{bmatrix} = G + P + S,
\] (32)
where \( \beta \) is a positive constant.

**Remark 3.** Note that if \( 0 < \beta, \mu < 1 \), by a similar using of Theorem 1, the GHSS-I and GHSS-II unconditionally converge to solution of \( Kx = b \). In addition, by some simple manipulations of Algorithm 1, the GHSS-I, GHSS-II and RGHSS can be used to solve Eq. (7).

As we can see in Eq. (7), to use our method in solving the proposed equation, we have to find the value of regularized parameter \( \mu \). Some existing methods for finding the value of \( \mu \) are the discrepancy principle method [29], the L-curve criterion method [30], and the generalized cross validation (GCV) method [31]. In our examples, we use the GCV method to find regularization parameter. The GCV method presumes that a missed data can be predicted by a good choice of pixel values. Being independent of knowledge about the noise variance is a property which makes this method applicable. In the GCV method,
the regularization parameter value is the value of \( \mu \), which minimizes the following GCV function:

\[
G(\mu) = \frac{\| A(\mu^T A + \mu^2 I)^{-1} A^T g - g \|_2}{(\text{trace}(I - A(\mu^T A + \mu^2 I)^{-1}))^2}.
\]

Finding the exact value of \( \mu \) from the GCV function is not economical in computation or sometimes even impossible. Therefore, we use the Kronecker product approximation to find an estimate value for \( \mu \). This method uses the singular value decomposition approximation to find the regularization parameter (see e.g., [14, 32, 33]).

To compare the original image with the restored one, the signal-to-noise ratio (SNR) and peak signal-to-noise ratio (PSNR) are defined as follows:

\[
\text{SNR} = 10 \log_{10} \frac{\|f_{\text{true}}\|_2^2}{\|f - f_{\text{true}}\|_2^2}, \quad \text{PSNR} = 10 \log_{10} \frac{4 \times 255^2 \times n^4}{\|f - f_{\text{true}}\|_2^2}
\]

where the size of the image is \( n \times n \) and \( f_{\text{true}}, f \) are original and restored images, respectively. For more investigation, we compare the relative error for the proposed methods. The relative error is considered as \( \frac{\|f - f_{\text{true}}\|_2}{\|f_{\text{true}}\|_2} \). The values of \( \alpha \) and \( \beta \) are approximately estimated by some tests on an image to obtain their optimal values in the GHSS, GHSS-I and GHSS-II methods. To find the optimal values of the RGHSS method, we use Theorem 3 and obtain the optimal value of \( \alpha \) for chosen \( \beta \). According to Theorem 3, to find the optimal values, the singular values of matrix \( A \) should be computed. Lv et al. in [20] used the Kronecker product approximation and found an approximation for the optimal value of \( \alpha \). Accordingly, first two matrices \( B_k \) and \( C_k \) are found, which minimize \( ||A - \sum B_k \otimes C_k||_F \), then the singular values of \( A \) are approximated by the small matrices \( B_k \) and \( C_k \). In all the examples, we use the same method to find the optimal value \( \alpha^* \) for chosen \( \beta \). Moreover, Algorithm 1 is implemented with \( \tau = 10^{-6} \). Furthermore, to solve the proposed system in Algorithm 1, the restarted GMRES method of MATLAB with \( \text{restart} = 15 \) and \( \zeta = 10^{-6} \) has been used.

**Example 1.** In this example, the cameraman grayscale image is used to investigate the proposed method. For degrading this image, the symmetric truncated Gaussian PSF is applied to the original image and 1% Gaussian white noise is added to the blurred image. By considering \( c \) as a normalization parameter, the proposed PSF can be formulated as follows:

\[
h_{ij} = \begin{cases} 
  ce^{-0.1(i^2+j^2)}, & \text{if } |i - j| \leq 8, \\
  0, & \text{otherwise}.
\end{cases}
\]

For this test, the true image is 256 \( \times \) 256 and the observed image domain is characterized by white lines. The true image, the Gaussian PSF and the blurred and noisy image are shown in Fig. 1. To implement the proposed methods to this test, the maximum number of outer iteration product in Algorithm 1 is \( M = 15 \). The values of PSNR and SNR of blurred and noisy image in this example are 26.06 and 14.21, respectively.

The given values of unknown parameters \((\beta, \alpha)\) for various methods in this example are available in Table 1. The PSNR, SNR and relative errors of the various methods are given in Tables 2-4. Moreover, the restored image by using the GHSS method for various BCs are
Table 1: Values of $(\beta, \alpha)$ for various methods in Example 1

<table>
<thead>
<tr>
<th>method</th>
<th>Zero</th>
<th>Periodic</th>
<th>Reflexive</th>
<th>Antireflexive</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>SHSS</td>
<td>(−0.33, −0.33)</td>
<td>(−0.33, −0.59)</td>
<td>(−0.88)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>GHSS-I</td>
<td>(−0.68, 0.68)</td>
<td>(−0.68, 0.67)</td>
<td>(−0.242)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>GHSS-II</td>
<td>(0.50, 0.67)</td>
<td>(0.50, 0.67)</td>
<td>(0.50, 0.67)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>GHSS</td>
<td>(0.65, 0.85)</td>
<td>(0.80, 0.05)</td>
<td>(0.20, 0.04)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RGHSS</td>
<td>(2.10, 0.40)</td>
<td>(2.50, 0.42)</td>
<td>(0.05, 0.07)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: PSNR of the various methods for Example 1

<table>
<thead>
<tr>
<th>method</th>
<th>Zero</th>
<th>Periodic</th>
<th>Reflexive</th>
<th>Antireflexive</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>SHSS</td>
<td>21.50</td>
<td>24.32</td>
<td>28.10</td>
<td>27.99</td>
<td>27.88</td>
</tr>
<tr>
<td>GHSS-I</td>
<td>21.98</td>
<td>24.94</td>
<td>27.94</td>
<td>28.03</td>
<td>27.49</td>
</tr>
<tr>
<td>GHSS-II</td>
<td>22.02</td>
<td>24.95</td>
<td>27.98</td>
<td>28.03</td>
<td>27.46</td>
</tr>
<tr>
<td>GHSS</td>
<td>22.47</td>
<td>25.17</td>
<td>28.74</td>
<td>28.95</td>
<td>28.98</td>
</tr>
<tr>
<td>RGHSS</td>
<td>22.03</td>
<td>24.95</td>
<td>28.67</td>
<td>28.90</td>
<td>28.97</td>
</tr>
</tbody>
</table>

Table 3: SNR of the various methods for Example 1

<table>
<thead>
<tr>
<th>method</th>
<th>Zero</th>
<th>Periodic</th>
<th>Reflexive</th>
<th>Antireflexive</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>GHSS-I</td>
<td>10.12</td>
<td>13.09</td>
<td>16.10</td>
<td>16.17</td>
<td>15.64</td>
</tr>
<tr>
<td>GHSS-II</td>
<td>10.17</td>
<td>13.10</td>
<td>16.13</td>
<td>16.17</td>
<td>15.61</td>
</tr>
<tr>
<td>GHSS</td>
<td>10.61</td>
<td>13.32</td>
<td>16.89</td>
<td>17.09</td>
<td>17.13</td>
</tr>
<tr>
<td>RGHSS</td>
<td>10.17</td>
<td>13.10</td>
<td>16.82</td>
<td>17.05</td>
<td>17.12</td>
</tr>
</tbody>
</table>

Table 4: Relative error of the various methods for Example 1

<table>
<thead>
<tr>
<th>method</th>
<th>Zero</th>
<th>Periodic</th>
<th>Reflexive</th>
<th>Antireflexive</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>SHSS</td>
<td>0.3294</td>
<td>0.2381</td>
<td>0.1541</td>
<td>0.1559</td>
<td>0.1579</td>
</tr>
<tr>
<td>GHSS-I</td>
<td>0.3118</td>
<td>0.2215</td>
<td>0.1568</td>
<td>0.1553</td>
<td>0.1652</td>
</tr>
<tr>
<td>GHSS-II</td>
<td>0.3101</td>
<td>0.2213</td>
<td>0.1561</td>
<td>0.1553</td>
<td>0.1658</td>
</tr>
<tr>
<td>GHSS</td>
<td>0.2945</td>
<td>0.2159</td>
<td>0.1431</td>
<td>0.1397</td>
<td>0.1391</td>
</tr>
<tr>
<td>RGHSS</td>
<td>0.3100</td>
<td>0.2213</td>
<td>0.1442</td>
<td>0.1404</td>
<td>0.1393</td>
</tr>
</tbody>
</table>
Figure 1: True image (left), the Gaussian PSF (center) and blurred and noisy image (right) for Example 1

<table>
<thead>
<tr>
<th>Method \ BC</th>
<th>Zero</th>
<th>Periodic</th>
<th>Reflexive</th>
<th>Antireflexive</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>SHSS</td>
<td>(–0.34)</td>
<td>(–0.33)</td>
<td>(–0.34)</td>
<td>(–0.60)</td>
<td>(–0.86)</td>
</tr>
<tr>
<td>GHSS-I</td>
<td>(–0.64)</td>
<td>(–0.64)</td>
<td>(–0.64)</td>
<td>(–0.64)</td>
<td>(–0.24)</td>
</tr>
<tr>
<td>GHSS-II</td>
<td>(0.50, 0.64)</td>
<td>(0.50, 0.65)</td>
<td>(0.50, 0.64)</td>
<td>(0.55, 0.64)</td>
<td>(0.50, 0.24)</td>
</tr>
<tr>
<td>GHSS</td>
<td>(0.15, 0.11)</td>
<td>(0.30, 0.14)</td>
<td>(0.30, 0.20)</td>
<td>(0.42, 0.09)</td>
<td>(0.26, 0.11)</td>
</tr>
<tr>
<td>RGHSS</td>
<td>(0.15, 0.08)</td>
<td>(0.12, 0.16)</td>
<td>(0.10, 0.12)</td>
<td>(0.17, 0.06)</td>
<td>(0.18, 0.07)</td>
</tr>
</tbody>
</table>

Table 5: Values of $(\beta, \alpha)$ for various methods in Example 2

Example 2. In this example, the $128 \times 128 \times 27$ simulated MRI of a human brain is considered. The proposed image is available in the MATLAB Image Processing Toolbox. An out-of-focus PSF using the function \texttt{psfDefocus} with \texttt{dim} = 9 and \texttt{R} = 4 in [8] is applied to blur the image. Moreover, a 2\% Gaussian noise is added to blurred image to obtain blurred and noisy image. The true and blurred and noisy images are shown in Fig. 3. In this test, the values of PSNR and SNR of the blurred and noisy image are 32.40 and 10.02, respectively. The value of maximum number of outer iteration product in Algorithm 1 is also considered $M = 15$.

The chosen values of $(\beta, \alpha)$ are presented in Table 5. The values of PSNR, SNR and relative errors are given in Tables 6-8. As the numerical results show, the RGHSS and GHSS methods are effective to restore image. The restored images with GHSS method for various BCs are shown in the Fig. 4 for further investigation.

Example 3. In this example, an astronomical object from a ground-based telescope is shown in Fig. 2. As the results show, the new methods, specially the RGHSS and GHSS methods are very effective and accurate and the accuracy of the RGHSS is close to that of the GHSS method.

Table 6: PSNR of the various methods for Example 2

<table>
<thead>
<tr>
<th>Method \ BC</th>
<th>Zero</th>
<th>Periodic</th>
<th>Reflexive</th>
<th>Antireflexive</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>SHSS</td>
<td>35.01</td>
<td>35.16</td>
<td>35.00</td>
<td>34.57</td>
<td>33.82</td>
</tr>
<tr>
<td>GHSS-I</td>
<td>34.76</td>
<td>34.86</td>
<td>34.75</td>
<td>34.71</td>
<td>33.40</td>
</tr>
<tr>
<td>GHSS-II</td>
<td>34.75</td>
<td>34.85</td>
<td>34.75</td>
<td>34.70</td>
<td>33.39</td>
</tr>
<tr>
<td>GHSS</td>
<td>35.77</td>
<td>35.95</td>
<td>35.70</td>
<td>35.65</td>
<td>34.54</td>
</tr>
<tr>
<td>RGHSS</td>
<td>35.77</td>
<td>35.94</td>
<td>35.72</td>
<td>35.65</td>
<td>34.54</td>
</tr>
</tbody>
</table>
Figure 2: Restored image with GHSS method for various BCs in Example 1. Up: Zero (left), periodic (center) and reflexive (right), down: antireflexive (left) and mean (right)

Table 7: SNR of the various methods for Example 2

<table>
<thead>
<tr>
<th>method \ BC</th>
<th>Zero</th>
<th>Periodic</th>
<th>Reflexive</th>
<th>Antireflexive</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>SHSS</td>
<td>12.63</td>
<td>12.78</td>
<td>12.62</td>
<td>12.19</td>
<td>11.44</td>
</tr>
<tr>
<td>GHSS-I</td>
<td>12.38</td>
<td>12.48</td>
<td>12.37</td>
<td>12.32</td>
<td>11.02</td>
</tr>
<tr>
<td>GHSS-II</td>
<td>12.37</td>
<td>12.47</td>
<td>12.36</td>
<td>12.32</td>
<td>11.44</td>
</tr>
<tr>
<td>GHSS</td>
<td>13.39</td>
<td>13.56</td>
<td>13.32</td>
<td>13.27</td>
<td>12.16</td>
</tr>
<tr>
<td>RGHSS</td>
<td>13.39</td>
<td>13.56</td>
<td>13.34</td>
<td>13.27</td>
<td>12.16</td>
</tr>
</tbody>
</table>

considered. The Keck telescope PSF is considered to blur the image. A 6% Gaussian noise is also added to the blurred image to obtain blurred and noisy image. The true image, the telescope PSF and the blurred and noisy images are given in Fig. 5. Now, we consider the proposed methods and compare them for reflexive BC. The number of outer iteration in all methods is supposed as $M = 20$. The restored images for various methods with reflexive BC are given in Fig. 6. To obtain this figure, the assumed values for $(\beta, \alpha)$ are given in Table 9. Moreover, the relative error of the proposed methods is given in this table. Now, we let $e_k = g - Af_k$ to compute the residual error in $k$th outer iteration of Algorithm 1. In Fig. 7, the convergence speed of our introduced methods for periodic and reflexive BCs are considered by plotting the $\|e_k\|_2/\|e_0\|_2$ versus the outer production number $k$. To plot this figure, we use the approximated optimal values for unknown iteration parameters in the proposed iterative methods. As we can see, the convergence of the RGHSS method is approximately as fast as the GHSS one while in the RGHSS method only one unknown parameter has been determined and therefore it is economical to use.
Table 8: Relative error of the various methods for Example 2

<table>
<thead>
<tr>
<th>method \ BC</th>
<th>Zero</th>
<th>Periodic</th>
<th>Reflexive</th>
<th>Antireflexive</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>SHSS</td>
<td>0.2337</td>
<td>0.2297</td>
<td>0.2339</td>
<td>0.2457</td>
<td>0.2679</td>
</tr>
<tr>
<td>GHSS-I</td>
<td>0.2406</td>
<td>0.2378</td>
<td>0.2407</td>
<td>0.2420</td>
<td>0.2812</td>
</tr>
<tr>
<td>GHSS-II</td>
<td>0.2407</td>
<td>0.2379</td>
<td>0.2409</td>
<td>0.2421</td>
<td>0.2817</td>
</tr>
<tr>
<td>GHSS</td>
<td>0.2141</td>
<td>0.2098</td>
<td>0.2157</td>
<td>0.2171</td>
<td>0.2467</td>
</tr>
<tr>
<td>RGHSS</td>
<td>0.2141</td>
<td>0.2098</td>
<td>0.2154</td>
<td>0.2171</td>
<td>0.2467</td>
</tr>
</tbody>
</table>

Figure 3: True image (left) and blurred and noisy image (right) for Example 2

5. Conclusion

In this paper, image restoration problem was considered for using five BCs: zero, periodic, reflexive, antireflexive and mean. This problem was reduced to solve a non-Hermitian positive definite linear system. We presented a new splitting for the coefficient matrix to use in GHSS method. A restricted version of the GHSS method was also considered. For this case, a theorem was given to investigate the convergence of the method and to find the optimal parameters in restricted version. The numerical result was also offered. As the numerical results show, our method is accurate and effective in image restoration problem.

Acknowledgments

We would like to thank Xiao-Guang Lv for his help in programming of this work. Moreover, the authors are grateful to anonymous referee and editor of the journal for their valuable comments and suggestions.

References

Figure 4: Restored image with GHSS for various BCs in Example 2. Up: Zero (left), periodic (center) and reflexive (right), down: antireflexive (left) and mean (right)


Figure 5: True image (left), the telescope PSF (center) and blurred and noisy image (right) for Example 3


Figure 6: Restored image for various methods with reflexive BC in Example 3. Up: SHSS (left), GHSS-I (center) and GHSS-II (right), down: GHSS (left) and RGHSS (right)


Figure 7: Comparison between residual errors for periodic (left) and reflexive (right) BC in Example 3


