On the solution of the fully fuzzy Sylvester matrix equation

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Received: date / Accepted: date

Abstract The aim of this paper is to propose a method for solving fully fuzzy Sylvester equation (FFSE) $AX+XB=C$, where A, B and C are fuzzy matrices. By using of α-cutting, FFSE is transformed to a generalized Sylvester matrix equation, and then to a crisp system of linear equations. Existence and uniqueness of a solution to this system are investigated. Two numerical examples are given to illustrate the proposed method.

Keywords fuzzy Sylvester equation, system of linear equations, trapezoidal fuzzy number, M-matrix.

Mathematics Subject Classification (2000) 15A39, 90C25

1 Introduction

Solving systems of linear equations is one of the main problems of science and engineering. In many applications, some of the system parameters are represented by fuzzy numbers rather than crisp. Therefore, it is needed to develop the available methods to a fuzzy linear system of equations (hereafter it is denoted by FLSE). Friedman et al. in [1] proposed a general model for solving an FLSE in which the entries of coefficient matrix are crisp numbers and its right-hand side vector is a fuzzy vector. They presented some conditions for the existence and uniqueness of a fuzzy solution to FLSE by using the embedding method [1] and converted the original system to a crisp system of linear equations.

It is well known that the Sylvester matrix equation is of the form

$$AX + XB = C, \tag{1}$$

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where $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$ are given matrices and $X \in \mathbb{R}^{m\times n}$ is unknown. Eq. (1) plays an important role in control theory, signal processing, model reduction, image processing, decoupling techniques for ODEs, implementation of implicit numerical methods for ODEs, and matrix block-diagonalization (for example, see [2–4, 6, 7] and references therein).

Salkuyeh in [8] developed the idea of [1] to the fuzzy Sylvester equation (FSS) $AX+XB = C$, where A and B are crisp matrices and C is a fuzzy matrix. Recently, He et al. in [5] proposed a new method for solving the same problem. Their method is more efficient than the method of [8]. In this paper, we consider a more general case where the matrices A, B and C are all fuzzy matrices. Hereafter, we call it fully fuzzy Sylvester equation (FFSE). In fact, by using the concept of α-cut, Eq. (1) is converted to a system of linear equations with crisp data and then the existence and uniqueness of a solution to the problem are investigated.

For convenience, some notations, definitions and results that will be used in the following parts are given in continuation. Throughout this paper, the vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ is said to be positive (nonnegative), if $x_i > 0$ ($x_i \geq 0$), $i = 1, 2, \ldots, n$. In this case, we write $x > 0$ ($x \geq 0$). Similar definitions can be written for matrices.

Definition 1 ([9]) If $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, then the Kronecker product of A and B is defined as the matrix $A \otimes B = (a_{ij}B) \in \mathbb{R}^{mp\times nq}$.

Definition 2 Assume that $Z = (z_1, \ldots, z_m) \in \mathbb{R}^{n\times m}$ where $z_i \in \mathbb{R}^n$, $i = 1, \ldots, n$, are the columns of Z. Then $vec(Z)$ is defined to be the (mn)-vector defined as $vec(Z) = (z_1^T, \ldots, z_m^T)^T$.

Theorem 1 *([9]) If $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{r\times s}$, $X \in \mathbb{R}^{n\times r}$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^s$, then*

$$(A \otimes B)(x \otimes y) = (Ax) \otimes (By),$$
$$vec(AXB) = (B^T \otimes A)vec(X).$$

Definition 3 ([10]) A matrix $A = (a_{ij})$ is said to be an M-matrix if $a_{ii} > 0$ for $i = 1, \ldots, n$, $a_{ij} \leq 0$, for $i \neq j$, A is nonsingular and $A^{-1} \geq 0$.

Lemma 1 *([11, Lemma 6.4]) A matrix $A = (a_{ij})$ with $a_{ij} \leq 0$, $i \neq j$, is an M-matrix if and only if there exists a positive vector x, such that Ax is positive.*

Theorem 2 *([10, Theorem 1.33]) Assume that A and B are two matrices that satisfy*

1. *$A \leq B$,*
2. *$b_{ij} \leq 0$, for all $i \neq j$.*

Then, if A is an M-matrix, so is the matrix B.

This paper is organized as follows. In Section 2, we review some basic concepts of the fuzzy numbers. Main results to solve fuzzy Sylvester equation are introduced in Section 3. Numerical results of the proposed method are presented in Section 4. Concluding remarks are given in Section 5.

2 Preliminaries

In this section, we review some concepts, lemmas and theorems concerning fuzzy numbers.

Definition 4 ([12]) A fuzzy number u is a fuzzy set with membership function $\mu_u : R \to [0, 1]$ that satisfies the following conditions

1. The function $\mu_u(x)$ is upper semi-continuous,
2. $\mu_u(x)$ is zero outside an interval $[c, d]$,
3. There are real numbers a, b, c and d such that $c \leq a \leq b \leq d$ and

(a) The function $\mu_u(x)$ is monotonic increasing on $[c, a]$,
(b) The function $\mu_u(x)$ is monotonic decreasing on $[b, d]$
(c) The function $\mu_u(x) = 1$ for $a \le x \le b$.

The set of all of fuzzy numbers is denoted by $\mathbb{E}$.

Definition 5 ([13]) A fuzzy number u is called nonnegative (positive) if its membership function is such that $\mu_u(x)$ is zero for all $x < 0$ ($x \le 0$).

Definition 6 ([13]) The α-cut of a fuzzy set u is the following crisp set

$$[u]_\alpha = \{\ x \mid \mu_u(x) \ge \alpha,\ \ \alpha \in [0, 1]\}.$$

Lemma 2 ([14]) The α-cut of a fuzzy number is a closed interval.

Definition 7 ([12,15]) The parametric form of a fuzzy number u is a pair of the form $[\underline{u}(\alpha), \overline{u}(\alpha)]$, $0 \le \alpha \le 1$ satisfying the following conditions:

1. the function $\underline{u}(\alpha)$ is a bounded monotonic increasing left continuous function,
2. the function $\overline{u}(\alpha)$ is a bounded monotonic decreasing left continuous function,
3. $\underline{u}(\alpha) \le \overline{u}(\alpha)$, $0 \le \alpha \le 1$.

A matrix A is said to be a fuzzy matrix if its entries are fuzzy numbers. The parametric form of a fuzzy matrix A is represented by $[\underline{A}(\alpha), \overline{A}(\alpha)]$ for $0 \le \alpha \le 1$.

Definition 8 Assume that u and v are two fuzzy sets. Then $u = v$ if and only if $[\tilde{u}]_\alpha = [\tilde{v}]_\alpha$ for any $0 \le \alpha \le 1$.

The next definition introduces the concept of "*property M*" for a fuzzy matrix.

Definition 9 We say that the n-by-n fuzzy matrix $A = (a_{ij})$ has "property M" if for any $0 \le \alpha \le 1$ and crisp matrix B satisfying $\underline{A}(\alpha) \le B \le \overline{A}(\alpha)$ it can be deduced that the matrix B is an M-matrix.

The next theorem provides a simple criterion to verify whether a fuzzy matrix has property M or not.

Theorem 3 *The matrix $A = (a_{ij})$ has property M if and only if $\underline{A}(0)$ and $\overline{A}(1)$ are M-matrices.*

Proof. Obviously, if A has property M, then $\underline{A}(0)$ and $\overline{A}(1)$ are M-matrices. Conversely, let $\underline{A}(\alpha) \le B \le \overline{A}(\alpha)$. Then, we see that

$$\underline{A}(0) \le \underline{A}(\alpha) \le B \le \overline{A}(\alpha) \le \overline{A}(1)$$

Since $\underline{A}(0)$ and $\overline{A}(1)$ are M-matrices, their diagonal and off-diagonal entries are positive and nonpositive, respectively. Now, from Theorem 2 and the fact that $\underline{A}(0) \le B$ we conclude that the matrix B is an M-matrix. Therefore, A has property M. □

Now we are ready to present our main results concerning fully fuzzy Sylvester matrix equation.

3 Main results

We consider Eq. (1) where A and B have property M and C is a fuzzy matrix. We focus our attention to verifying existence and seeking a nonnegative solution $X = (x_{ij})$ to the fully fuzzy Sylvester equation, which means that $x_{ij} \geq 0$, for all $i, j = 1, \ldots, n$. Let $A^{(1)} = (a_{ij}^{(1)})$, $A^{(2)} = (a_{ij}^{(2)})$, $B^{(1)} = (b_{ij}^{(1)})$ and $B^{(2)} = (b_{ij}^{(2)})$ where

$$a_{ij}^{(1)} = \begin{cases} a_{ij}, & a_{ij} > 0, \\ 0, & \text{otherwise}, \end{cases} \qquad a_{ij}^{(2)} = \begin{cases} 0, & a_{ij} > 0, \\ a_{ij}, & \text{otherwise}, \end{cases}$$

and

$$b_{ij}^{(1)} = \begin{cases} b_{ij}, & b_{ij} > 0, \\ 0, & \text{otherwise}, \end{cases} \qquad b_{ij}^{(2)} = \begin{cases} 0, & b_{ij} > 0, \\ b_{ij}, & \text{otherwise}. \end{cases}$$

Then we have

$$A = A^{(1)} + A^{(2)} \quad \text{and} \quad B = B^{(1)} + B^{(2)}.$$

We now rewrite the relation (1) in the form

$$\begin{aligned} \underline{AX}(\alpha) + \underline{XB}(\alpha) &= \underline{C}(\alpha), \\ \overline{AX}(\alpha) + \overline{XB}(\alpha) &= \overline{C}(\alpha). \end{aligned} \tag{2}$$

Since X is nonnegative, Eq. (2) is modified as follows

$$\begin{aligned} \underline{A^{(1)}}(\alpha)\ \underline{X}(\alpha) + \overline{A^{(2)}}(\alpha)\ \overline{X}(\alpha) + \underline{X}(\alpha)\ \underline{B^{(1)}}(\alpha) + \overline{X}(\alpha)\ \overline{B^{(2)}}(\alpha) &= \underline{C}(\alpha), \\ \overline{A^{(1)}}(\alpha)\ \overline{X}(\alpha) + \underline{A^{(2)}}(\alpha)\ \underline{X}(\alpha) + \overline{X}(\alpha)\ \overline{B^{(1)}}(\alpha) + \underline{X}(\alpha)\ \underline{B^{(2)}}(\alpha) &= \overline{C}(\alpha). \end{aligned} \tag{3}$$

By using the "*vec*" operator this system can be transformed to the system of linear equations

$$\mathcal{A}(\alpha)\mathcal{X}(\alpha) = \mathcal{C}(\alpha), \tag{4}$$

where

$$\mathcal{A}(\alpha) = \begin{pmatrix} I_n \otimes \underline{A^{(1)}}(\alpha) + \underline{B^{(1)}}(\alpha)^T \otimes I_m & I_n \otimes \overline{A^{(2)}}(\alpha) + \overline{B^{(2)}}(\alpha)^T \otimes I_m \\ I_n \otimes \underline{A^{(2)}}(\alpha) + \underline{B^{(2)}}(\alpha)^T \otimes I_m & I_n \otimes \overline{A^{(1)}}(\alpha) + \overline{B^{(1)}}(\alpha)^T \otimes I_m \end{pmatrix}, \tag{5}$$

and

$$\mathcal{C}(\alpha) = \begin{pmatrix} vec(\underline{C}(\alpha)) \\ vec(\overline{C}(\alpha)) \end{pmatrix}, \qquad \mathcal{X}(\alpha) = \begin{pmatrix} vec(\underline{X}(\alpha)) \\ vec(\overline{X}(\alpha)) \end{pmatrix}.$$

Theorem 4 *Assume that the fuzzy matrices A and B have property M. Then, the matrix $\mathcal{A}(\alpha)$ defined in Eq. (5) is an M-matrix for any $0 \leq \alpha \leq 1$.*

Proof. If the fuzzy matrices A and B have property M, then obviously $\underline{A}(\alpha)$, $\overline{A}(\alpha)$, $\underline{B}(\alpha)$ and $\overline{B}(\alpha)$ are M-matrices. Since $\underline{A}(\alpha)$ is an M-matrix, $\underline{A}(\alpha)^T$ is an M-matrix, too. Therefore, according to Lemma 1 there exists a positive vector x such that $\underline{A}(\alpha)^T x > 0$. On the other hand, again by Lemma 1 there exists a positive vector y such $\underline{B}(\alpha)y > 0$, since $\underline{B}(\alpha)$ is an M-matrix. Obviously, we have $\overline{A}(\alpha)^T x > 0$ and $\overline{B}(\alpha)y > 0$, since we have $\underline{A}(\alpha) \leq \overline{A}(\alpha)$ and $\underline{B}(\alpha) \leq \overline{B}(\alpha)$. We show that the matrix $\mathcal{A}(\alpha)^T$ is an M-matrix. It is not difficult to see that the diagonal and off-diagonal entries of $\mathcal{A}(\alpha)$ are positive and nonpositive, respectively. We now set

$$\mathcal{Z} = \begin{pmatrix} y \otimes x \\ y \otimes x \end{pmatrix}.$$

Obviously, we have $\mathcal{Z} > 0$. Then we see that

$$
\begin{aligned}
\mathcal{A}^T(\alpha)\mathcal{Z} &= \begin{pmatrix} y \otimes \underline{A^{(1)}}(\alpha)^T x + \underline{B^{(1)}}(\alpha) y \otimes x + y \otimes \underline{A^{(2)}}(\alpha)^T x + \underline{B^{(2)}}(\alpha) y \otimes x \\ y \otimes \overline{A^{(2)}}(\alpha)^T x + \underline{B^{(2)}}(\alpha) y \otimes x + y \otimes \overline{A^{(1)}}(\alpha)^T x + \overline{B^{(1)}}(\alpha) y \otimes x \end{pmatrix} \\
&= \begin{pmatrix} y \otimes \underline{A}(\alpha)^T x + \underline{B}(\alpha) y \otimes x \\ y \otimes \overline{A}(\alpha)^T x + \underline{B}(\alpha) y \otimes x \end{pmatrix} \\
&> 0.
\end{aligned}
$$

This shows that the matrix $\mathcal{A}(\alpha)$ is an M-matrix for every $\alpha \in [0, 1]$. □

Now, assume that all of the assumptions of Theorem 4 hold. Then $\mathcal{A}(\alpha)$ is an M-matrix for every $0 \leq \alpha \leq 1$, and as a result we have $\mathcal{A}(\alpha)^{-1} \geq 0$. Hence, if $C \geq 0$ then according to Eq. (4) we see that this system has a unique solution of the form

$$\mathcal{X}(\alpha) = \mathcal{A}(\alpha)^{-1}\mathcal{C}(\alpha) \geq 0.$$

This shows that the corresponding fully fuzzy Sylvester system has a unique nonnegative solution. However, the solution is not necessary fuzzy in general.

4 Numerical examples

In this section, two examples are presented to illustrate the theoretical results of the previous section. In both of the examples the entries of the matrices A, B and C are chosen triangular fuzzy numbers.

Example 1 We consider the fuzzy matrices A, B, and C where their α-cuts are given as follows

$$
\begin{aligned}
[A]_\alpha &= \begin{pmatrix} (2.9 + 0.1\alpha, 3.2 + 0.2\alpha) & (-2.1 + 0.1\alpha, -1.9 + 0.1\alpha) \\ (-2.1 + 0.1\alpha, -1.9 + 0.1\alpha) & (3.9 + 0.1\alpha, 4.2 + 0.2\alpha) \end{pmatrix}, \\
[B]_\alpha &= \begin{pmatrix} (2.99 + 0.01\alpha, 3.02 - 0.02\alpha) & (-2.01 + 0.01\alpha, -1.99 - 0.01\alpha) \\ (-1.01 + 0.01\alpha, -0.99 - 0.01\alpha) & (1.99 + 0.01\alpha, 2.02 - 0.02\alpha) \end{pmatrix}, \\
[C]_\alpha &= \begin{pmatrix} (3.5 + 1.5\alpha, 7 - 2\alpha) & (5 + 2\alpha, 10 - 3\alpha) \\ (6 + 2\alpha, 11 - 3\alpha) & (-1 + 3\alpha, 4 - 2\alpha) \end{pmatrix}.
\end{aligned}
$$

By the proposed method the matrices $\mathcal{A}(\alpha)$ and $\mathcal{C}(\alpha)$ in Eq. (4) are given by

$$
\begin{aligned}
\mathcal{A}(\alpha) &= \begin{pmatrix} \mathcal{A}_{11}(\alpha) & \mathcal{A}_{12}(\alpha) \\ \mathcal{A}_{21}(\alpha) & \mathcal{A}_{22}(\alpha) \end{pmatrix}, \\
\mathcal{C}(\alpha) &= \begin{pmatrix} 3.5 + 1.5r & 6 + 2r & 5 + 2r & -1 + 3r & 7 - 2r & 11 - 3r & 10 - 3r & 4 - 2r \end{pmatrix}^T
\end{aligned}
$$

in which

$$
\begin{aligned}
\mathcal{A}_{11}(\alpha) &= \begin{pmatrix} 5.89 + 0.11\alpha & 0 & 0 & 0 \\ 0 & 6.89 + 0.11\alpha & 0 & 0 \\ 0 & 0 & 4.89 + 0.11\alpha & 0 \\ 0 & 0 & 0 & 5.89 + 0.11\alpha \end{pmatrix}, \\
\mathcal{A}_{12}(\alpha) &= \begin{pmatrix} 0 & -1.9 - 0.1\alpha & -0.99 - 0.01\alpha & 0 \\ -1.9 - 0.1\alpha & 0 & 0 & -0.99 - 0.01\alpha \\ -1.99 - 0.01\alpha & 0 & 0 & -1.9 - 0.1\alpha \\ 0 & -1.99 - 0.01\alpha & -1.9 - 0.1\alpha & 0 \end{pmatrix}, \\
\mathcal{A}_{21}(\alpha) &= \begin{pmatrix} 0 & -2.1 + 0.1\alpha & -1.01 + 0.01\alpha & 0 \\ -2.1 + 0.1\alpha & 0 & 0 & -1.01 + 0.01\alpha \\ -2.01 + 0.01\alpha & 0 & 0 & -2.1 + 0.1\alpha \\ 0 & -2.01 + 0.01\alpha & -2.1 + 0.1\alpha & 0 \end{pmatrix},
\end{aligned}
$$

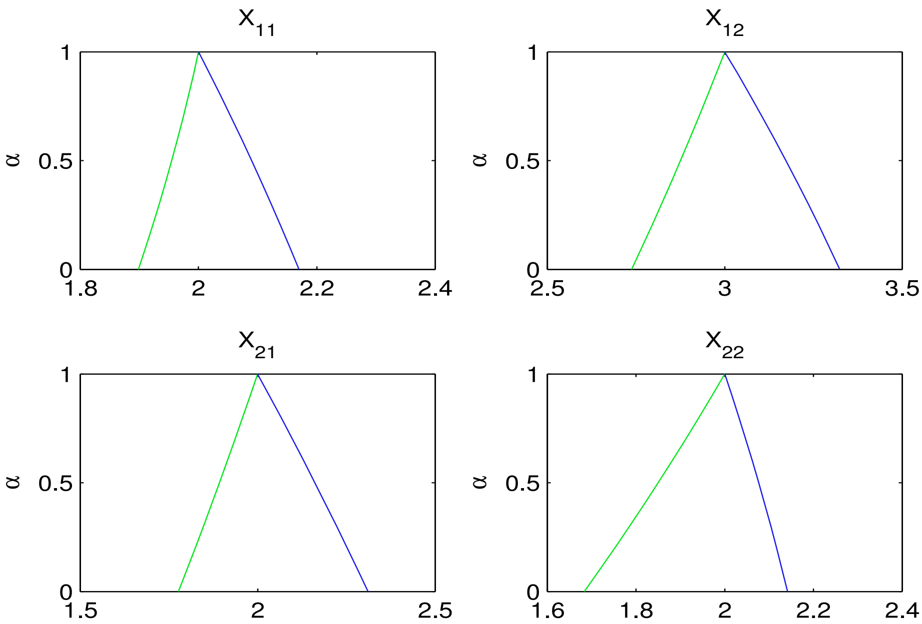

Fig. 1 The computed solution for Example 1.

$$\mathcal{A}_{22}(\alpha) = \begin{pmatrix} 6.22-0.22\alpha & 0 & 0 & 0 \\ 0 & 7.22-0.22\alpha & 0 & 0 \\ 0 & 0 & 5.22-0.22\alpha & 0 \\ 0 & 0 & 0 & 6.22-0.22\alpha \end{pmatrix}.$$

It is easy to verify that for every $\alpha \in [0,1]$, the matrix $\mathcal{A}(\alpha)$ is an M-matrix. The entries of computed solution X are displayed in Figure 1. As seen the entries of the computed solution are all fuzzy numbers. As we mentioned in the previous section it is not necessary the computed solution to be fuzzy. Another observation which can be posed here is that the entries of the computed solution are all nonnegative.

Example 2 In this example, we consider the fuzzy matrices A, B and C with α-cuts

$$[A]_\alpha = \begin{pmatrix} (3.9+0.1r, 4.2-0.2r) & (-1.9+0.9r, -0.6-0.4r) & (-0.1+0.1r, 0.1-0.1r) \\ (-1.1+0.1r, -0.9-0.1r) & (4.8+0.2r, 5.2-0.2r) & (-0.1+0.1r, 0.2-0.2r) \\ (-0.1+0.1r, 0.1-0.1r) & (-0.1+0.1r, 0.1-0.1r) & (3.8+0.2r, 4.2-0.2r) \end{pmatrix},$$

$$[B]_\alpha = \begin{pmatrix} (3.9+0.1r, 4.2-0.2r) & (-0.6+0.6r, 0.9-0.9r) & (-1.9+0.9r, -0.3-0.7r) \\ (-0.1+0.1r, 0.1-0.1r) & (3.9+0.1r, 4.2-0.2r) & (-1.1+0.1r, -0.9-0.1r) \\ (-1.1+0.1r, -0.9-0.1r) & (-1.2+0.2r, -0.9-0.1r) & (4.8+0.2r, 5.3-0.3r) \end{pmatrix},$$

$$[C]_\alpha = \begin{pmatrix} (1+2r, 5-2r) & (3+r, 5-r) & (3+r, 5-r) \\ (2+r, 4-r) & (3+2r, 7-2r) & (4+r, 6-r) \\ (4+r, 6-r) & (1+r, 3-r) & (r, 2-r) \end{pmatrix}.$$

As the previous example we have computed the solution X. The entries of the computed have presented in Figure 2. As we see the computed solution is fuzzy and nonnegative.

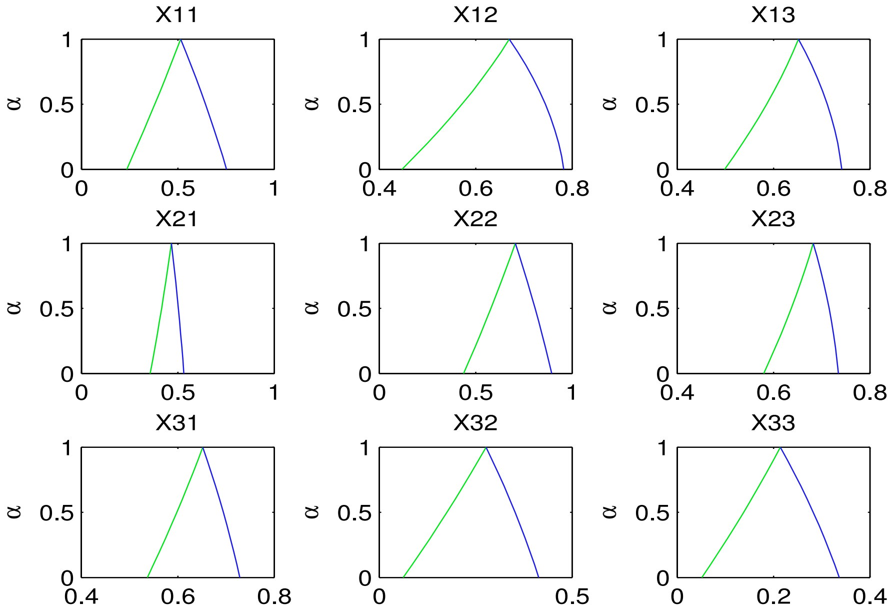

Fig. 2 The computed solution for Example 2.

5 Conclusion

We have considered the fully fuzzy Sylvester matrix equation $AX + XB = C$ where A, B and C are all fuzzy matrices. By using the concept of α-cut of a fuzzy number we have transformed the corresponding system to a crisp system of linear equations such that its entries are all functions of α where $0 \leq \alpha \leq 1$. Some theoretical results concerning the existence of a nonnegative solution have been given. Two numerical examples have been given to illustrate the theoretical results.

Acknowledgement

The work the first, second and fourth authors is partially supported by University of Guilan.

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