A FINITE ITERATIVE ALGORITHM FOR HERMITIAN REFLEXIVE AND SKEW-HERMITIAN SOLUTION GROUPS OF THE GENERAL COUPLED LINEAR MATRIX EQUATIONS

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Abstract. In this paper, we focus on the following coupled linear matrix equations

$$\mathcal{M}_i(X,Y) = \mathcal{M}_{i1}(X) + \mathcal{M}_{i2}(Y) = L_i,$$

with

$$\mathcal{M}_{i\ell}(W) = \sum_{j=1}^{q}\left(\sum_{\lambda=1}^{t_1^{(\ell)}} A_{ij\lambda}^{(\ell)} W_j B_{ij\lambda}^{(\ell)} + \sum_{\mu=1}^{t_2^{(\ell)}} C_{ij\mu}^{(\ell)} \overline{W}_j D_{ij\mu}^{(\ell)} + \sum_{\nu=1}^{t_3^{(\ell)}} E_{ij\nu}^{(\ell)} W_j^T F_{ij\nu}^{(\ell)}\right), \quad \ell = 1,2.$$

where $A_{ij\lambda}^{(\ell)}, B_{ij\lambda}^{(\ell)}$, $C_{ij\mu}^{(\ell)}, D_{ij\mu}^{(\ell)}$, $E_{ij\nu}^{(\ell)}, F_{ij\nu}^{(\ell)}$ and L_i (for $i \in I[1,p]$) are given matrices with appropriate dimensions defined over complex number field. Our object is to obtain the solution groups $X = (X_1, X_2, \ldots, X_q)$ and $Y = (Y_1, Y_2, \ldots, Y_q)$ of the considered coupled linear matrix equations such that X and Y are the groups of the Hermitian reflexive and skew-Hermitian matrices, respectively. To do so, an iterative algorithm is proposed which stops within finite number of steps in the exact arithmetic. Moreover, the algorithm determines the solvability of the mentioned coupled linear matrix equations over the Hermitian reflexive and skew-Hermitian matrices, automatically. In the case that the coupled linear matrix equations are consistent, the least-norm Hermitian reflexive and skew-Hermitian solution groups can be computed by choosing suitable initial iterative matrix groups. In addition, the unique optimal approximate Hermitian reflexive and skew-Hermitian solution groups to given arbitrary matrix groups are derived. Finally, some numerical experiments are reported to illustrate the validity of our established theoretical results and feasibly of the presented algorithm.

Keywords: Linear matrix equation; Iterative algorithm; Hermitian reflexive matrix; Skew-Hermitian matrix.
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1. Introduction

We first present some notations which are employed throughout this paper. The symbols $\mathrm{tr}(A), A^T, \overline{A}, A^H$, $R(A)$, $N(A)$ are exploited to indicate the trace, the transpose, the conjugate, the conjugate transpose, the column space and the null space of the matrix A, respectively. Furthermore, $\mathbb{C}^{m\times n}$ denotes the set of all $m\times n$ complex matrices. The set of all symmetric orthogonal matrices in $\mathbb{C}^{n\times n}$ is shown by $\mathbb{SOC}^{n\times n}$, i.e., $P \in \mathbb{SOC}^{n\times n}$ if and only if $P = P^H = P^{-1}$. An $n\times n$ matrix Y is said to be skew-Hermitian if $Y^H = -Y$. The set of all $n\times n$ skew-Hermitian matrices is represented by $\mathbb{SH}^{n\times n}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called Hermitian

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reflexive with respect to P if $X = X^H = PXP$. The notation $\mathbb{HC}_r^{n\times n}(P)$ stands for the set of all $n \times n$ Hermitian reflexive matrices with respect to P.

The linear matrix equations play a cardinal role in several areas, such as control theory, system theory, stability theory, perturbation analysis and some other fields of pure and applied mathematics. Iterative algorithms have extensive applications in numerous fields such as iterative learning control and system identification, e.g., hierarchical gradient based and hierarchical least squares based iterative parameter identification for CARARMA; see [17, 18, 27]. In the literature, a large number of papers are devoted to applying different kinds of iterative algorithm for solving various linear matrix equations, for more details see [2, 3, 4, 6, 12, 13, 14, 15, 16, 20, 23, 24, 30, 31, 36, 38, 40, 41, 42] and the references therein.

In [11], the authors have presented two iterative algorithms for computing the Hermitian reflexive and skew-Hermitian solutions of the Sylvester matrix equation $AX + XB = C$.

In [32], the following coupled Sylvester-transpose matrix equations

$$\sum_{\eta=1}^{p} \left(A_{i\eta}X_\eta B_{i\eta} + C_{i\eta}X_\eta^T D_{i\eta}\right) = F_i \qquad i = 1, 2, \ldots, N, \tag{1.1}$$

have been considered where $A_{i\eta}, B_{i\eta}, C_{i\eta}, D_{i\eta}, F_i$ for $i \in I[1, N]$ and $\eta \in I[1, p]$ are given known matrices with suitable dimensions defined over real numbers field, X_η for $\eta \in I[1, p]$ are the matrices to be determined. In the case that (1.1) has a unique solution, the authors have proposed an iterative algorithm by using the hierarchical identification principle.

Recently, the idea of conjugate gradient (CG) [29] method has been developed for constructing iterative algorithms to compute the solutions of different kinds of linear matrix equations over generalized reflexive and anti-reflexive, generalized bisymmetric, generalized centro-symmetric, mirror-symmetric, skew-symmetric and (P, Q)-reflexive matrices, for more details see [3, 8, 9, 10, 19, 20, 21, 24, 25, 26, 34, 37, 39]. For instance, Wu et al. [37] have considered the following coupled Sylvester-conjugate matrix equations

$$\sum_{\eta=1}^{p} \left(A_{i\eta}X_\eta B_{i\eta} + C_{i\eta}\overline{X_\eta} D_{i\eta}\right) = F_i, \qquad i = 1, 2, \ldots, p, \tag{1.2}$$

and proposed an iterative algorithm where $A_{i\eta}, B_{i\eta}, C_{i\eta}$, $D_{i\eta}$ and F_i, $i, \eta \in I[1, p]$, are given matrices with appropriate sizes defined over complex number field.

Lately, Song et al. [33] have developed the CG method to solve the coupled linear matrix equation (1.1). In [28], the authors have offered an efficient iterative algorithm for solving the next matrix equation

$$\sum_{i=1}^{s} A_i V + \sum_{j=1}^{t} B_j W = \sum_{l=1}^{m} E_l \bar{V} F_l + C,$$

by extending the idea of the CG method. In [22], Jiang and Li have utilized the idea of the CG method to elaborate a finite iterative algorithm to resolve the subsequent quaternion matrix equation

$$\sum_{l=1}^{u} A_l X B_l + \sum_{s=1}^{v} C_s \bar{X} D_s = F,$$

over (P, Q)-reflexive matrices.

In [39], the authors have focused on the coupled matrix equations

$$(AXB - CYD, EXF - GYH) = (M, N). \tag{1.3}$$

By developing the idea of CG method, an iterative algorithm has been constructed for computing generalized reflexive (anti-reflexive) solution of Eq. (1.3). Wu et al. [35, 36] have presented two different algorithms, by using the idea of gradient based [29] and CG methods, for solving the following matrix equation

$$\sum_{l=1}^{s_1} A_l X B_l + \sum_{l=1}^{s_2} C_l \overline{X} D_l + \sum_{l=1}^{s_3} G_l X^T H_l + \sum_{l=1}^{s_4} M_l X^H N_l = F, \tag{1.4}$$

where $A_l \in \mathbb{C}^{m\times r}, B_l \in \mathbb{C}^{s\times n}, l \in I[1, s_1]$, $C_l \in \mathbb{C}^{m\times r}, D_l \in \mathbb{C}^{s\times n}, l \in I[1, s_2]$, $G_l \in \mathbb{C}^{m\times s}, H_l \in \mathbb{C}^{r\times n}, l \in I[1, s_3]$, $M_l \in \mathbb{C}^{m\times s}$, $N_l \in \mathbb{C}^{r\times n}, l \in I[1, s_4]$, and $F \in \mathbb{C}^{m\times n}$ are given known matrices and $X \in \mathbb{C}^{r\times s}$ is the matrix to be determined.

In [42, 43, 44], Zhou et al. have proposed gradient based algorithms to solve some kinds of (coupled) matrix equations. In these works, it has been assumed that the mentioned problem has a unique solution. Necessary and sufficient conditions are established under which the examined algorithms are convergent. In this work, we consider an entirely general class of coupled matrix equations which are not necessarily uniquely solvable and offer an iterative algorithm to solve our considered coupled matrix equations over Hermitian reflexive and skew-Hermitian matrices. The properties of our algorithm are investigated. It is demonstrated that the algorithm is convergent within finite number of steps in the absence of round off errors.

Suppose that

$$\mathcal{C} := \mathbb{C}^{n_1\times n_1} \times \cdots \times \mathbb{C}^{n_q\times n_q}, \quad \mathcal{L} := \mathbb{C}^{r_1\times s_1} \times \cdots \times \mathbb{C}^{r_p\times s_p},$$

and define the linear operator as follows:

$$\begin{aligned} \mathcal{M}_\kappa : \mathcal{C} &\to \mathcal{L} \\ W &\to \mathcal{M}_\kappa(W) \end{aligned}$$

such that $W = (W_1, W_2, \ldots, W_q)$, $\mathcal{M}_\ell(W) = (M_{1\ell}(W), M_{2\ell}(W), \ldots, M_{p\ell}(W))$ and

$$\mathcal{M}_{i\ell}(W) = \sum_{j=1}^{q} \left(\sum_{\lambda=1}^{t_1^{(\ell)}} A_{ij\lambda}^{(\ell)} W_j B_{ij\lambda}^{(\ell)} + \sum_{\mu=1}^{t_2^{(\ell)}} C_{ij\mu}^{(\ell)} \overline{W}_j D_{ij\mu}^{(\ell)} + \sum_{\nu=1}^{t_3^{(\ell)}} E_{ij\nu}^{(\ell)} W_j^T F_{ij\nu}^{(\ell)} \right), \quad \ell = 1, 2.$$

where the matrices $A_{ij\lambda}^{(\ell)}, C_{ij\mu}^{(\ell)}, E_{ijk}^{(\ell)} \in \mathbb{C}^{r_i\times n_j}$ $B_{ij\lambda}^{(\ell)}, D_{ij\mu}^{(\ell)}, F_{ij\nu}^{(\ell)} \in \mathbb{C}^{n_j\times s_i}$ and $L_i \in \mathbb{C}^{r_i\times s_i}$ are given.

To the best of our knowledge, the solution of the following general form of the coupled Sylvester-conjugate and transpose matrix equations have not been investigated hitherto,

$$\mathcal{M}_1(X) = L, \tag{1.5}$$

where the matrix group $L = (L_1, L_2, \ldots, L_p)$ is given and $X = (X_1, X_2, \ldots, X_q)$ is the group of unknown matrices with $X_j \in \mathbb{C}^{n_j\times n_j}$, $j \in I[1, q]$. More precisely, the problem of finding the solution groups of Hermitian reflexive (shew-Hermitian) matrices for the coupled linear matrix equations (1.5) has not been mentioned so far.

Consider the following linear operator

$$\begin{aligned} \mathcal{M} : \mathcal{C} \times \mathcal{C} &\to \mathcal{L} \\ (X, Y) &\to \mathcal{M}(X, Y), \end{aligned}$$

and the general coupled linear matrix equations

$$\mathcal{M}(X, Y) = \mathcal{M}_1(X) + \mathcal{M}_2(Y) = L, \tag{1.6}$$

where the matrix group $L = (L_1, L_2, \ldots, L_p)$ are given in which $L_i \in \mathbb{C}^{r_i\times s_i}$ for $i \in I[1, p]$. The unknown solution groups $X = (X_1, X_2, \ldots, X_q)$ and $Y = (Y_1, Y_2, \ldots, Y_q)$ are computed such that $X_j \in \mathbb{HC}_r^{n_j\times n_j}(P_j)$ and $Y_j \in \mathbb{SH}^{n_j\times n_j}$ where the matrices $P_j \in \mathbb{SOR}^{n_j\times n_j}$ are given for $j \in I[1, q]$.

Remark 1.1. *Note that the linear coupled matrix equations* (1.6) *contain numerous of the investigated linear matrix equations such as Eqs.* (1.1), (1.2), (1.3) *and* (1.4). *Also, we would like to point out that the presented results can be employed for solving of the coupled linear matrix equations of the form* (1.5) *over Hermitian reflexive (skew-Hermitian) matrices solely.*

The following inner product and its induced norm will be utilized throughout this paper.

Definition 1.2. *Suppose that* $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_k)$ *and* $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_k)$ *where* $\Phi_i, \Psi_i \in \mathbb{C}^{r_i \times s_i}$ *for* $i = 1, 2, \ldots, k$. *We define the inner product* $\langle ., . \rangle$ *as follows:*

$$\langle \Phi, \Psi \rangle := \mathrm{Re}[\sum_{i=1}^{k} \mathrm{tr}(\Phi_j^H \Psi_j)], \tag{1.7}$$

where $\mathrm{Re}[Z]$ *stands for the real part of complex number* Z.

Remark 1.3. *For* $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_k)$, *where* $\Phi_i \in \mathbb{C}^{r_i \times s_i}$ *for* $i = 1, 2, \ldots, k$, *the norm of* Φ *is defined by* $\|\Phi\|^2 := \mathrm{Re}[\sum_{i=1}^{k} \mathrm{tr}(\Phi_j^H \Phi_j)] = \sum_{j=1}^{k} \mathrm{tr}(\Phi_j^H \Phi_j)$.

1.1. **Problem reformulation.** The main purpose of this paper is to focus on the solution of the following problems.

Problem 1. *For given matrices* $A_{ij\lambda}^{(\ell)}, C_{ij\mu}^{(\ell)}, E_{ij\nu}^{(\ell)} \in \mathbb{C}^{r_i \times n_j}$, $B_{ij\lambda}^{(\ell)}, D_{ij\mu}^{(\ell)}, F_{ij\nu}^{(\ell)} \in \mathbb{C}^{n_j \times s_i}$, $L_i \in \mathbb{C}^{r_i \times s_i}$ *and* $P_j \in \mathbb{SOC}^{n_j \times n_j}$, *find the matrix groups*

$$X = (X_1, X_2, \ldots, X_q) \quad \textit{and} \quad Y = (Y_1, Y_2, \ldots, Y_q),$$

such that $X_j \in \mathbb{HC}_r^{n_j \times n_j}(P_j)$, $Y_j \in \mathbb{SH}^{n_j \times n_j}$ *and satisfy Eq.* (1.6) *where* $i \in I[1, p]$, $j \in I[1, q]$, $\lambda \in I[1, t_1^{(\ell)}]$, $\mu \in I[1, t_2^{(\ell)}]$, $\nu \in I[1, t_3^{(\ell)}]$ *and* $\ell = 1, 2$.

Problem 2. *Suppose that Problem 1 is consistent and* $S_{\mathcal{M}}$ *denotes the set of its solution groups. Assume that the matrix groups* $\Gamma_x = (\Gamma_{1x}, \Gamma_{2x}, \ldots, \Gamma_{qx})$ *and* $\Gamma_y = (\Gamma_{1y}, \Gamma_{2y}, \ldots, \Gamma_{qy})$ *are given, where* $\Gamma_{jx} \in \mathbb{HC}_r^{n_j \times n_j}(P_j)$ *and* $\Gamma_{jy} \in \mathbb{SH}^{n_j \times n_j}$ *for* $j \in I[1, q]$. *Find the solution groups* $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_q)$ *and* $\widetilde{Y} = (\widetilde{Y}_1, \widetilde{Y}_2, \ldots, \widetilde{Y}_q)$ *of* (1.6) *such that*

$$\left\|\Gamma_x - \widetilde{X}\right\|^2 + \left\|\Gamma_y - \widetilde{Y}\right\|^2 = \min_{(X,Y) \in S_{\mathcal{M}}} \{\|\Gamma_x - X\|^2 + \|\Gamma_y - Y\|^2\},$$

where $\widetilde{X}_j \in \mathbb{HC}_r^{n_j \times n_j}(P_j)$ *and* $\widetilde{Y}_j \in \mathbb{SH}^{n_j \times n_j}$ *for* $j \in I[1, q]$.

In fact, Problem 2 gives the optimal approximate solution groups $(\widetilde{X}, \widetilde{Y})$ to the given matrix groups (Γ_x, Γ_y) where $\widetilde{X}$ and Γ_x ($\widetilde{Y}$ and Γ_y) are the groups of Hermitian reflexive (skew-Hermitian) matrices.

1.2. **Definitions and properties.** In this subsection, we recall some necessary concepts which are used in the next sections.

Assume that $A = [a_{ij}]_{m \times s}$ and $B = [b_{ij}]_{n \times q}$ defined over complex (real) number field, the Kronecker product of the matrices A and B is defined as the $mn \times sq$ matrix $A \otimes B = [a_{ij} B]$. The "vec" operator transforms a matrix A of size $m \times s$ to a vector $a = \mathrm{vec}(A)$ of size $ms \times 1$ by stacking the columns of A. In this paper, the following relation is utilized (See [5])

$$\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X).$$

Lemma 1.4. *Let* $X \in \mathbb{R}^{m \times n}$ *be an arbitrary matrix. Then*

$$\mathrm{vec}(X^T) = P(m, n)\mathrm{vec}(X),$$

where $P(m, n)$ *is uniquely determined by the integers* m *and* n.

Proof. See [7]. □

We may conclude the next remark from Lemma 1.4 immediately.

Remark 1.5. *Let $X \in \mathbb{R}^{m\times n}$ be an arbitrary matrix. Then,*

$$\mathrm{vec}(X^H) = \mathrm{vec}(\overline{X}^T) = P(m,n)\mathrm{vec}(\overline{X}),$$

where $P(m,n)$ is uniquely determined by the integers m and n.

Some properties of the matrix $P(m,n)$ are given as follows ([1, 7, 24]):

1. For two arbitrary integers m and n, $P(m,n)$ has the ensuing explicit form

$$P(m,n) = \begin{pmatrix} E_{11}^T & E_{12}^T & \cdots & E_{1n}^T \\ E_{21}^T & E_{22}^T & \cdots & E_{2n}^T \\ \vdots & \vdots & \ddots & \vdots \\ E_{m1}^T & E_{m2}^T & \cdots & E_{mn}^T \end{pmatrix}_{mn\times mn},$$

where each E_{ij} is an $m \times n$ matrix with the entry at position (i,j) being one and the others being zero, $i \in I[1,m]$ and $j \in I[1,n]$.

2. For two arbitrary integers m and n, $P(m,n)$ is an orthogonal matrix, i.e.,
$P(m,n)P^T(m,n) = P^T(m,n)P(m,n) = I_{mn}$.
3. For two arbitrary integers m and n, $P(m,n) = P^T(n,m)$.

The rest of this paper is organized as follows. In Section 2, we introduce useful linear operators. Some properties of these linear operators are also established which are efficient for presenting and analyzing our algorithm. In Section 3, we propose an iterative algorithm to obtain the solutions of Problem 1. It is theoretically shown that if Problem 1 is consistent, then the proposed algorithm converges within finite number of steps in the absence of roundoff errors. In addition, an approach is offered for solving Problem 2. Section 4 is devoted to presenting some numerical examples to demonstrate the feasibly of our propounded algorithm to resolve Problems 1 and 2. Finally, the paper is ended with a brief conclusion in Section 5.

2. Useful linear operators

Throughout this paper, we employ some efficient linear operators. In this section, we introduce these linear operators and establish some of their properties and their relationships.
Let us define the linear operators $\mathcal{A}_1$ and $\mathcal{A}_2$ as follows:

$$\begin{aligned} \mathcal{A}_\ell : \mathcal{L} &\to \mathcal{C} \\ Z &\to \mathcal{A}_\ell(Z), \end{aligned}$$

where $Z = (Z_1, Z_2, \ldots, Z_p)$, $\mathcal{A}_\ell(Z) = (A_{1\ell}(Z), A_{2\ell}(Z), \ldots, A_{q\ell}(Z))$ and

$$A_{j\ell}(Z) = \sum_{i=1}^{p}\Big(\sum_{\lambda=1}^{t_1^{(\ell)}} (A_{ij\lambda}^{(\ell)})^H Z_i (B_{ij\lambda}^{(\ell)})^H + \sum_{\mu=1}^{t_2^{(\ell)}} (\overline{C_{ij\mu}^{(\ell)}})^H \overline{Z}_i (\overline{D_{ij\mu}^{(\ell)}})^H + \sum_{\nu=1}^{t_3^{(\ell)}} \overline{F_{ij\nu}^{(\ell)}} Z_i^T \overline{E_{ij\nu}^{(\ell)}}\ \Big),$$

for $j \in I[1,q]$ and $\ell = 1,2$.

Furthermore, we present the linear operators $\mathcal{D}_1$ and $\mathcal{D}_2$ such that

$$\begin{aligned} \mathcal{D}_\ell : \mathcal{L} &\to \mathcal{C} \\ Z &\to \mathcal{D}_\ell(Z), \end{aligned}$$

where $\mathcal{D}_\ell(Z) = (D_{1\ell}(Z), D_{2\ell}(Z), \ldots, D_{q\ell}(Z))$ with

$$D_{j1}(Z) = \tfrac{1}{4}\left(A_{j1}(Z) + (A_{j1}(Z))^H + P_jA_{j1}(Z)P_j + P_j(A_{j1}(Z))^HP_j\right),$$

$$D_{j2}(Z) = \tfrac{1}{2}\left(A_{j2}(Z) - (A_{j2}(Z))^H\right),$$

in which the matrices $P_j \in \mathbb{SOC}^{n_j\times n_j}$ for $j \in I[1,q]$ are given.

Proposition 2.1. *Assume that* $X = (X_1, X_2, \ldots, X_q)$, $Y = (Y_1, Y_2, \ldots, Y_q)$, $Z = (Z_1, Z_2, \ldots, Z_p)$, *and the matrices* $P_j \in \mathbb{SOC}^{n_j\times n_j}$ *for* $j \in I[1,q]$ *are given. For* $i \in I[1,p]$ *and* $j \in I[1,q]$,

(*a*) *if* $X_j \in \mathbb{HC}_r^{n_j\times n_j}(P_j)$ *and* $Z_i \in \mathbb{C}^{r_i\times s_i}$, *then*

$$\langle \mathcal{A}_1(Z), X\rangle = \langle \mathcal{D}_1(Z), X\rangle .$$

(*b*) *if* $Y_j \in \mathbb{SH}^{n_j\times n_j}$ *and* $Z_i \in \mathbb{C}^{r_i\times s_i}$,*then*

$$\langle \mathcal{A}_2(Z), Y\rangle = \langle \mathcal{D}_2(Z), Y\rangle .$$

Proof. Assume that $X = (X_1, X_2, \ldots, X_q)$ and $Y = (Y_1, Y_2, \ldots, Y_q)$ are the groups of Hermitian and skew-Hermitian matrices, respectively, that is $X_j \in \mathbb{HC}_r^{n_j\times n_j}(P_j)$ and $Y_j \in \mathbb{SH}^{n_j\times n_j}$ for $j \in I[1,q]$. Moreover, let $Z = (Z_1, Z_2, \ldots, Z_p)$ be an arbitrary group of matrices such that $Z_i \in \mathbb{C}^{r_i\times s_i}$, $i \in I[1,p]$. As $X_j = X_j^H$ and $Y_j = -Y_j^H$, we deduce that

$$\langle \mathcal{A}_1(Z), X\rangle = \mathrm{Re}[\textstyle\sum_{j=1}^{q} \left((A_{j1}(Z))^H X_j\right)] = \mathrm{Re}[\textstyle\sum_{j=1}^{q} \left((A_{j1}(Z))^H X_j^H\right)],$$

$$-\langle \mathcal{A}_2(Z), Y\rangle = \mathrm{Re}[\textstyle\sum_{j=1}^{q} \left(-(A_{j2}(Z))^H Y_j\right)] = \mathrm{Re}[\textstyle\sum_{j=1}^{q} \left((A_{j2}(Z))^H Y_j^H\right)].$$

For two arbitrary matrices A and B, it is well-known that $\mathrm{Re}[\mathrm{tr}(A^H B^H)] = \mathrm{Re}[\mathrm{tr}(AB)]$. Hence, we get

$$\langle \mathcal{A}_1(Z), X\rangle = \langle \mathcal{B}_1(Z), X\rangle ,$$

$$-\langle \mathcal{A}_2(Z), Y\rangle = \langle \mathcal{B}_2(Z), Y\rangle ,$$

where $\mathcal{B}_\ell(Z) = ((A_{1\ell}(Z))^H, (A_{2\ell}(Z))^H, \ldots, (A_{q\ell}(Z))^H)$ for $\ell = 1, 2$. Now, we may conclude the results immediately. □

Proposition 2.2. *Consider the arbitrary matrix groups* $X = (X_1, X_2, \ldots, X_q)$, $Y = (Y_1, Y_2, \ldots, Y_q)$, $Z = (Z_1, Z_2, \ldots, Z_p)$ *where* $X_j, Y_j \in \mathbb{C}^{n_j\times n_j}$, $Z_i \in \mathbb{C}^{r_i\times s_i}$ *for* $i \in I[1,p]$ *and* $j \in I[1,q]$. *Then*

$$\langle \mathcal{M}(X,Y), Z\rangle = \langle X, \mathcal{A}_1(Z)\rangle + \langle Y, \mathcal{A}_2(Z)\rangle .$$

Proof. It is known that $\mathcal{M}(X,Y) = \mathcal{M}_1(X) + \mathcal{M}_2(Y)$. Hence, we only need to prove that

$$\langle \mathcal{M}_1(X), Z\rangle = \langle X, \mathcal{A}_1(Z)\rangle \quad \text{and} \quad \langle \mathcal{M}_2(Y), Z\rangle = \langle Y, \mathcal{A}_2(Z)\rangle .$$

To this end, we show that for an arbitrary matrix group $W = (W_1, W_2, \ldots, W_q)$,

$$\langle \mathcal{M}_\ell(W), Z\rangle = \langle W, \mathcal{A}_\ell(Z)\rangle , \quad \ell = 1, 2.$$

By some easy computations, we derive:

$$\langle \mathcal{M}_\ell(W), Z\rangle = \sum_{i=1}^{p} \mathrm{Re}\,[tr\,(\,(\sum_{j=1}^{q}(\sum_{\lambda=1}^{t_1^{(\ell)}}(B_{ij\lambda}^{(\ell)})^H W_j^H (A_{ij\lambda}^{(\ell)})^H + \sum_{\mu=1}^{t_2^{(\ell)}}(D_{ij\mu}^{(\ell)})^H \overline{W}_j^H (C_{ij\mu}^{(\ell)})^H$$
$$+\sum_{\nu=1}^{t_3^{(\ell)}}(F_{ij\nu}^{(\ell)})^H (W_j^T)^H (G_{ij\nu}^{(\ell)})^H)\,)\,Z_i)], \quad \ell = 1, 2.$$

It is known that for two arbitrary matrices A and B, $\mathrm{Re}[\mathrm{tr}(AB)] = \mathrm{Re}[\mathrm{tr}(\overline{AB})] = \mathrm{Re}[\mathrm{tr}(BA)] = \mathrm{Re}[\mathrm{tr}(A^T B^T)] = \mathrm{Re}[\mathrm{tr}(A^H B^H)]$. Hence,

$$\langle \mathcal{M}_\ell(W), Z\rangle = \sum_{i=1}^{p}\sum_{j=1}^{q} \mathrm{Re}[\sum_{\lambda=1}^{t_1^{(\ell)}} \mathrm{tr}(W_j^H (A_{ij\lambda}^{(\ell)})^H Z_i (B_{ij\lambda}^{(\ell)})^H)] + \mathrm{Re}[\sum_{\mu=1}^{t_2^{(\ell)}} \mathrm{tr}(\overline{W}_j^H (C_{ij\mu}^{(\ell)})^H Z_i (D_{ij\mu}^{(\ell)})^H)]$$
$$+\mathrm{Re}[\sum_{\nu=1}^{t_3^{(\ell)}} \mathrm{tr}(W_j^H \overline{F_{ij\nu}^{(\ell)}} Z_i^T \overline{G_{ij\nu}^{(\ell)}})],$$

$$= \sum_{j=1}^{q} \mathrm{Re}[\mathrm{tr}(W_j^H(\sum_{i=1}^{p}(\sum_{\lambda=1}^{t_1^{(\ell)}}(A_{ij\lambda}^{(\ell)})^H Z_i (B_{ij\lambda}^{(\ell)})^H + \sum_{\mu=1}^{t_2^{(\ell)}}(\overline{C_{ij\mu}^{(\ell)}})^H \overline{Z}_i (\overline{D_{ij\mu}^{(\ell)}})^H$$
$$+ \sum_{\nu=1}^{t_3^{(\ell)}} \overline{F_{ij\nu}^{(\ell)}} Z_i^T \overline{G_{ij\nu}^{(\ell)}})))] = \langle W, \mathcal{A}_\ell(Z)\rangle, \qquad \ell = 1, 2,$$

which completes the proof. □

For simplicity, we define the linear operators $\mathfrak{M}_1$ and $\mathfrak{M}_2$ as follows:

$$\mathfrak{M}_\ell : \mathcal{C} \to \mathcal{L}$$
$$W \to \mathfrak{M}_\ell(W),$$

where $\mathfrak{M}_\ell(W) = (\mathfrak{M}_{1\ell}(W), \mathfrak{M}_{2\ell}(W), \ldots, \mathfrak{M}_{p\ell}(W))$ and for $\ell = 1, 2$,

$$\mathfrak{M}_{i\ell}(W) = \sum_{j=1}^{q}\left(\sum_{\lambda=1}^{t_1^{(\ell)}}\left(B_{ij\lambda}^{(\ell)}\right)^H W_j \left(A_{ij\lambda}^{(\ell)}\right)^H + \sum_{\mu=1}^{t_2^{(\ell)}}\left(D_{ij\mu}^{(\ell)}\right)^H \overline{W}_j \left(C_{ij\mu}^{(\ell)}\right)^H + \sum_{\nu=1}^{t_3^{(\ell)}}\left(F_{ij\nu}^{(\ell)}\right)^H W_j^T \left(E_{ij\nu}^{(\ell)}\right)^H\right).$$

Proposition 2.3. *The coupled linear matrix equations (1.6) has the solution groups $X = (X_1, X_2, \ldots, X_q)$ and $Y = (Y_1, Y_2, \ldots, Y_q)$ such that $X_j \in \mathbb{HC}_r^{n_j\times n_j}$ and $Y_j \in \mathbb{SH}^{n_j\times n_j}$ for $j \in I[1,q]$ if and only if the following system of matrix equations are consistent*

$$\text{(2.1)} \qquad \begin{cases} \mathcal{M}_1(X) + \mathcal{M}_2(Y) = L, \\ \mathfrak{M}_1(X) - \mathfrak{M}_2(Y) = L^{\mathcal{H}}, \\ \mathcal{M}_1(\widehat{X}) + \mathcal{M}_2(Y) = L, \\ \mathfrak{M}_1(\widehat{X}) - \mathfrak{M}_2(Y) = L^{\mathcal{H}}, \end{cases}$$

where $\widehat{X} = (P_1X_1P_1, P_2X_2P_2, \ldots, P_qX_qP_q)$, $L^{\mathcal{H}} = (L_1^H, L_2^H, \ldots, L_p^H)$ and the matrices $P_j \in \mathbb{SOR}^{n_j\times n_j}$ for $j \in I[1,q]$ are given.

Proof. Assume that the matrix groups $X = (X_1, X_2, \ldots, X_q)$ and $Y = (Y_1, Y_2, \ldots, Y_q)$ are the solution groups of (1.6) where $X_j \in \mathbb{HC}_r^{n_j\times n_j}(P_j)$ and $Z_j \in \mathbb{SH}^{n_j\times n_j}$ for $j \in I[1,q]$. Straightforward computations show that X and Y satisfy (2.1).

Conversely, suppose that $X = (X_1, X_2, \ldots, X_q)$ and $Y = (Y_1, Y_2, \ldots, Y_q)$ satisfy (2.1). We define the matrix groups $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_q)$ and $\widetilde{Y} = (\widetilde{Y}_1, \widetilde{Y}_2, \ldots, \widetilde{Y}_q)$, such that

$$\widetilde{X}_j = \frac{1}{4}\left(X_j + X_j^H + P_jX_jP_j + P_jX_j^HP_j\right) \quad \text{and} \quad \widetilde{Y}_j = \frac{1}{2}\left(Y_j - Y_j^H\right), \quad j = 1, 2, \ldots, q.$$

It is not difficult to see that $\widetilde{X}$ and $\widetilde{Y}$ are the groups of Hermitian reflexive and skew-Hermitian matrices, respectively, which gratify (1.6). □

3. An algorithm and its analysis

In this section, we first propose an algorithm for solving Problem 1. Then, an approach is presented for the solution of Problem 2.

3.1. **Presented algorithm.** In the following, the idea of the conjugate gradient method is utilized to construct an iterative algorithm for solving the general coupled linear matrix equations (1.6) over the mixed set of Hermitian reflexive and skew-Hermitian matrices.

Algorithm 1. (Proposed algorithm for solving Problem 1)

1. Input the matrices $A_{ij\lambda}^{(\ell)}, C_{ij\mu}^{(\ell)}, E_{ij\nu}^{(\ell)} \in \mathbb{C}^{r_i\times n_j}$, $B_{ij\lambda}^{(\ell)}, D_{ij\mu}^{(\ell)}, F_{ij\nu}^{(\ell)} \in \mathbb{C}^{n_j\times s_i}$, $L_i \in \mathbb{C}^{r_i\times s_i}$, $P_j \in \mathbb{SOC}^{n_j\times n_j}$, where i, j, λ, μ, ν and ℓ are integer numbers such that $i \in I[1,p], j \in I[1,q], \lambda \in I[1, t_1^{(\ell)}], \mu \in I[1, t_2^{(\ell)}], \nu \in I[1, t_3^{(\ell)}]$ and $\ell = 1, 2$.

2. Choose a tolerance ε and two arbitrary groups of matrices $X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \ldots, X_q^{(1)})$ and $Y^{(1)} = (Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_q^{(1)})$ such that $X_j^{(1)} \in \mathbb{HC}_r^{n_j\times n_j}(P_j)$ and $Y_j^{(1)} \in \mathbb{SH}^{n_j\times n_j}$ for $j = 1, 2, \ldots, q$.

3. Calculate $R^{(1)} = L - \mathcal{M}(X^{(1)}, Y^{(1)})$; Set $P_x^{(1)} = \mathcal{D}_1(R^{(1)})$, $P_y^{(1)} = \mathcal{D}_2(R^{(1)})$ and $k = 1$.

4. If "$\|R^{(k)}\| < \varepsilon$" or "$R^{(k)} \neq 0$ but $P_x^{(k)} = 0$ and $P_y^{(k)} = 0$" then Stop; Otherwise go to 5.

5. Compute

$$
\begin{aligned}
(X^{(k+1)}, Y^{(k+1)}) &= (X^{(k)}, Y^{(k)}) + \frac{\left\|R^{(k)}\right\|^2}{\left\|P_x^{(k)}\right\|^2 + \left\|P_y^{(k)}\right\|^2}\,(P_x^{(k)}, P_y^{(k)});\\
R^{(k+1)} &= R^{(k)} - \frac{\left\|R^{(k)}\right\|^2}{\left\|P_x^{(k)}\right\|^2 + \left\|P_y^{(k)}\right\|^2}\,\mathcal{M}(P_x^{(k)}, P_y^{(k)});\\
P_x^{(k+1)} &= \mathcal{D}_1(R^{(k+1)}) + \frac{\left\|R^{(k+1)}\right\|^2}{\left\|R^{(k)}\right\|^2} P_x^{(k)};\\
P_y^{(k+1)} &= \mathcal{D}_2(R^{(k+1)}) + \frac{\left\|R^{(k+1)}\right\|^2}{\left\|R^{(k)}\right\|^2} P_y^{(k)}.
\end{aligned}
$$

6. Go to Step 4.

Remark 3.1. *For the matrix groups* $X^{(k)} = (X_1^{(k)}, X_2^{(k)}, \ldots, X_q^{(k)})$, $Y^{(k)} = (Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_q^{(k)})$, $P_x^{(k)} = (P_{1x}^{(k)}, P_{2x}^{(k)}, \ldots, P_{qx}^{(k)})$ *and* $P_y^{(k)} = (P_{1y}^{(k)}, P_{2y}^{(k)}, \ldots, P_{qy}^{(k)})$, *produced by Algorithm 1, it is not difficult to show that* $X_j^{(k)}, P_{jx}^{(k)} \in \mathbb{HC}_r^{n_j\times n_j}(P_j)$ *and* $Y_j^{(k)}, P_{jy}^{(k)} \in \mathbb{SH}^{n_j\times n_j}$, $j \in I[1, q]$.

Lemma 3.2. *Assume that the sequences* $\{R^{(k)}\}$, $\{P_x^{(k)}\}$ *and* $\{P_y^{(k)}\}$, $k = 1, 2, \ldots, s$, $(R^{(k)} \neq 0$, $k \in I[1, s])$ *are produced by Algorithm 1. Then,*

$$
\left\langle R^{(i)}, R^{(j)} \right\rangle = 0 \text{ and } \left\langle P_x^{(i)}, P_x^{(j)} \right\rangle + \left\langle P_y^{(i)}, P_y^{(j)} \right\rangle = 0, \quad i, j = 1, 2, \ldots, s, \ (i \neq j). \tag{3.1}
$$

Proof. We only need to show that (3.1) is true for $1 \leq i < j \leq k$. To this end, we use the mathematical induction.

Step 1. For $k = 2$, Propositions 2.1 and 2.2 implies that

$$
\begin{aligned}
\left\langle R^{(1)}, R^{(2)} \right\rangle &= \left\|R^{(1)}\right\|^2 - \frac{\left\|R^{(1)}\right\|^2}{\left\|P_x^{(1)}\right\|^2 + \left\|P_y^{(1)}\right\|^2} \left\langle \mathcal{M}(P_x^{(1)}, P_y^{(1)}), R^{(1)} \right\rangle\\
&= \left\|R^{(1)}\right\|^2 - \frac{\left\|R^{(1)}\right\|^2}{\left\|P_x^{(1)}\right\|^2 + \left\|P_y^{(1)}\right\|^2} \left(\left\langle P_x^{(1)}, \mathcal{A}_1(R^{(1)}) \right\rangle + \left\langle P_y^{(1)}, \mathcal{A}_2(R^{(1)}) \right\rangle \right)\\
&= \left\|R^{(1)}\right\|^2 - \frac{\left\|R^{(1)}\right\|^2}{\left\|P_x^{(1)}\right\|^2 + \left\|P_y^{(1)}\right\|^2} \left(\left\langle P_x^{(1)}, \mathcal{D}_1(R^{(1)}) \right\rangle + \left\langle P_y^{(1)}, \mathcal{D}_2(R^{(1)}) \right\rangle \right)\\
&= \left\|R^{(1)}\right\|^2 - \frac{\left\|R^{(1)}\right\|^2}{\left\|P_x^{(1)}\right\|^2 + \left\|P_y^{(1)}\right\|^2} \left(\left\langle P_x^{(1)}, P_x^{(1)} \right\rangle + \left\langle P_y^{(1)}, P_y^{(1)} \right\rangle \right)\\
&= \left\|R^{(1)}\right\|^2 - \left\|R^{(1)}\right\|^2 = 0.
\end{aligned}
$$

Also, we have:

$$\begin{aligned}
\left\langle P_x^{(1)}, P_x^{(2)} \right\rangle + \left\langle P_y^{(1)}, P_y^{(2)} \right\rangle &= \left\langle P_x^{(1)}, \mathcal{D}_1(R^{(2)}) + \frac{\|R^{(2)}\|^2}{\|R^{(1)}\|^2} P_x^{(1)} \right\rangle \\
&+ \left\langle P_y^{(1)}, \mathcal{D}_2(R^{(2)}) + \frac{\|R^{(2)}\|^2}{\|R^{(1)}\|^2} P_y^{(1)} \right\rangle \\
&= \left\langle \mathcal{M}(P_x^{(1)}, P_y^{(1)}), R^{(2)} \right\rangle + \frac{\|R^{(2)}\|^2}{\|R^{(1)}\|^2} (\left\| P_x^{(1)} \right\|^2 + \left\| P_y^{(1)} \right\|^2) \\
&= \frac{\left\| P_x^{(1)} \right\|^2 + \left\| P_y^{(1)} \right\|^2}{\|R^{(1)}\|^2} \left\langle R^{(1)} - R^{(2)}, R^{(2)} \right\rangle \\
&+ \frac{\|R^{(2)}\|^2}{\|R^{(1)}\|^2} \left(\left\| P_x^{(1)} \right\|^2 + \left\| P_y^{(1)} \right\|^2 \right) = 0.
\end{aligned}$$

Step 2. Suppose that (3.1) holds when $k = \omega$, we show that the assertion is true for $k = \omega + 1$. Straightforward computations show that

$$\begin{aligned}
\left\langle R^{(\omega)}, R^{(\omega+1)} \right\rangle &= \left\| R^{(\omega)} \right\|^2 - \frac{\|R^{(\omega)}\|^2}{\left\| P_x^{(\omega)} \right\|^2 + \left\| P_y^{(\omega)} \right\|^2} \left\langle R^{(\omega)}, \mathcal{M}(P_x^{(\omega)}, P_y^{(\omega)}) \right\rangle \\
&= \left\| R^{(\omega)} \right\|^2 - \frac{\|R^{(\omega)}\|^2}{\left\| P_x^{(\omega)} \right\|^2 + \left\| P_y^{(\omega)} \right\|^2} \left(\left\langle \mathcal{D}_1(R^{(\omega)}), P_x^{(\omega)} \right\rangle + \left\langle \mathcal{D}_2(R^{(\omega)}), P_y^{(\omega)} \right\rangle \right) \\
&= \left\| R^{(\omega)} \right\|^2 - \frac{\|R^{(\omega)}\|^2}{\left\| P_x^{(\omega)} \right\|^2 + \left\| P_y^{(\omega)} \right\|^2} \left(\left\langle P_x^{(\omega)} - \frac{\|R^{(\omega)}\|^2}{\|R^{(\omega-1)}\|^2} P_x^{(\omega-1)}, P_x^{(\omega)} \right\rangle \right. \\
&+ \left. \left\langle P_y^{(\omega)} - \frac{\|R^{(\omega)}\|^2}{\|R^{(\omega-1)}\|^2} P_y^{(\omega-1)}, P_y^{(\omega)} \right\rangle \right) = 0,
\end{aligned}$$

and,

$$\begin{aligned}
\left\langle P_x^{(\omega)}, P_x^{(\omega+1)} \right\rangle + \left\langle P_y^{(\omega)}, P_y^{(\omega+1)} \right\rangle &= \left\langle P_x^{(\omega)}, \mathcal{D}_1(R^{(\omega+1)}) \right\rangle + \left\langle P_y^{(\omega)}, \mathcal{D}_2(R^{(\omega+1)}) \right\rangle \\
&+ \frac{\|R^{(\omega+1)}\|^2}{\|R^{(\omega)}\|^2} (\left\| P_x^{(\omega)} \right\|^2 + \left\| P_y^{(\omega)} \right\|^2) \\
&= \left\langle \mathcal{M}(P_x^{(\omega)}, P_y^{(\omega)}), R^{(\omega+1)} \right\rangle + \frac{\|R^{(\omega+1)}\|^2}{\|R^{(\omega)}\|^2} (\left\| P_x^{(\omega)} \right\|^2 + \left\| P_y^{(\omega)} \right\|^2) \\
&= \frac{\left\| P_x^{(\omega)} \right\|^2 + \left\| P_y^{(\omega)} \right\|^2}{\|R^{(\omega)}\|^2} \left\langle R^{(\omega)} - R^{(\omega+1)}, R^{(\omega+1)} \right\rangle \\
&+ \frac{\|R^{(\omega+1)}\|^2}{\|R^{(\omega)}\|^2} \left(\left\| P_x^{(\omega)} \right\|^2 + \left\| P_y^{(\omega)} \right\|^2 \right) = 0.
\end{aligned}$$

For $j = 1, 2, \ldots, \omega - 1$, we conclude that

$$\begin{aligned}
\left\langle R^{(j)}, R^{(\omega+1)} \right\rangle &= -\frac{\left\| R^{(\omega)} \right\|^2}{\left\| P_x^{(\omega)} \right\|^2 + \left\| P_y^{(\omega)} \right\|^2} \left\langle R^{(j)}, \mathcal{M}(P_x^{(\omega)}, P_y^{(\omega)}) \right\rangle \\
&= -\frac{\left\| R^{(\omega)} \right\|^2}{\left\| P_x^{(\omega)} \right\|^2 + \left\| P_y^{(\omega)} \right\|^2} \left(\left\langle \mathcal{D}_1(R^{(j)}), P_x^{(\omega)} \right\rangle + \left\langle \mathcal{D}_2(R^{(j)}), P_y^{(\omega)} \right\rangle \right) \\
&= -\frac{\left\| R^{(\omega)} \right\|^2}{\left\| P_x^{(\omega)} \right\|^2 + \left\| P_y^{(\omega)} \right\|^2} \left(\left\langle P_x^{(j)} - \frac{\left\| R^{(j)} \right\|^2}{\left\| R^{(j-1)} \right\|^2} P_x^{(j-1)}, P_x^{(\omega)} \right\rangle \right. \\
&+ \left. \left\langle P_y^{(j)} - \frac{\left\| R^{(j)} \right\|^2}{\left\| R^{(j-1)} \right\|^2} P_y^{(j-1)}, P_y^{(\omega)} \right\rangle \right) = 0,
\end{aligned}$$

and,

$$\begin{aligned}
\left\langle P_x^{(j)}, P_x^{(\omega+1)} \right\rangle + \left\langle P_y^{(j)}, P_y^{(\omega+1)} \right\rangle &= \left\langle P_x^{(j)}, \mathcal{D}_1(R^{(\omega)}) + \frac{\left\| R^{(\omega)} \right\|^2}{\left\| R^{(\omega-1)} \right\|^2} P_x^{(\omega)} \right\rangle \\
&+ \left\langle P_y^{(j)}, \mathcal{D}_2(R^{(\omega)}) + \frac{\left\| R^{(\omega)} \right\|^2}{\left\| R^{(\omega-1)} \right\|^2} P_y^{(\omega)} \right\rangle \\
&= \left\langle \mathcal{M}_1(P_x^{(j)}), R^{(\omega)} \right\rangle + \left\langle \mathcal{M}_2(P_y^{(j)}), R^{(\omega)} \right\rangle \\
&= \left\langle \mathcal{M}(P_x^{(j)}, P_y^{(j)}), R^{(\omega)} \right\rangle \\
&= \frac{\left\| P_x^{(j)} \right\|^2 + \left\| P_y^{(j)} \right\|^2}{\left\| R^{(j)} \right\|^2} \left\langle R^{(j)} - R^{(j-1)}, R^{(\omega)} \right\rangle = 0.
\end{aligned}$$

Now, by the principle of the mathematical induction the assertion (3.1) is true. □

Lemma 3.3. *Suppose that Problem 1 is consistent. Moreover, assume that* $X^* = (X_1^*, X_2^*, \ldots, X_q^*)$, $Y^* = (Y_1^*, Y_2^*, \ldots, Y_q^*)$ *are arbitrary solutions of Problem 1. Presume that* $X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \ldots, X_q^{(1)})$ *and* $Y^{(1)} = (Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_q^{(1)})$ *are given such that* $X_j^{(1)} \in \mathbb{HC}_r^{n_j \times n_j}(P_j)$ *and* $Y_j^{(1)} \in \mathbb{SH}^{n_j \times n_j}$ *for* $j \in I[1, q]$. *Then the sequences* $X^{(i)}$, $Y^{(i)}$, $R^{(i)}$, $P_x^{(i)}$ *and* $P_y^{(i)}$ *produced by Algorithm 1 satisfy the following relation.*

$$\left\langle P_x^{(i)}, X^* - X^{(i)} \right\rangle + \left\langle P_y^{(i)}, Y^* - Y^{(i)} \right\rangle = \| R^{(i)} \|^2 \quad i = 1, 2, 3, \ldots \quad . \tag{3.2}$$

Proof. We establish the conclusion (3.2) by the mathematical induction. For $i = 1$, we derive:

$$\begin{aligned}
\left\langle P_x^{(1)}, X^* - X^{(1)} \right\rangle + \left\langle P_y^{(1)}, Y^* - Y^{(1)} \right\rangle &= \left\langle \mathcal{D}_1(R^{(1)}), X^* - X^{(1)} \right\rangle + \left\langle \mathcal{D}_2(R^{(1)}), Y^* - Y^{(1)} \right\rangle \\
&= \left\langle \mathcal{A}_1(R^{(1)}), X^* - X^{(1)} \right\rangle + \left\langle \mathcal{A}_2(R^{(1)}), Y^* - Y^{(1)} \right\rangle \\
&= \left\langle R^{(1)}, \mathcal{M}_1(X^* - X^{(1)}) \right\rangle + \left\langle R^{(1)}, \mathcal{M}_2(Y^* - Y^{(1)}) \right\rangle \\
&= \left\langle R^{(1)}, L - \mathcal{M}(X^{(1)}, Y^{(1)}) \right\rangle \\
&= \left\| R^{(1)} \right\|^2.
\end{aligned}$$

Suppose the assertion (3.2) is valid for $i = \omega$. Then, for $i = \omega + 1$ we deduce that

$$\begin{aligned}
\left\langle P_x^{(\omega+1)}, X^* - X^{(\omega+1)} \right\rangle \ &+ \ \left\langle P_y^{(\omega+1)}, Y^* - Y^{(\omega+1)} \right\rangle = \left\langle \mathcal{D}_1(R^{(\omega+1)}), X^* - X^{(\omega+1)} \right\rangle \\
&+ \ \frac{\|R^{(\omega+1)}\|^2}{\|R^{(\omega)}\|^2} \left\langle P_x^{(\omega)}, X^* - X^{(\omega+1)} \right\rangle \\
&+ \ \left\langle \mathcal{D}_2(R^{(\omega+1)}), X^* - X^{(\omega+1)} \right\rangle \\
&+ \ \frac{\|R^{(\omega+1)}\|^2}{\|R^{(\omega)}\|^2} \left\langle P_y^{(\omega)}, Y^* - Y^{(\omega+1)} \right\rangle \\
&= \ \left\langle R^{(\omega+1)}, \mathcal{M}_1(X^* - X^{(\omega+1)}) \right\rangle \\
&+ \ \frac{\|R^{(\omega+1)}\|^2}{\|R^{(\omega)}\|^2} \left\langle P_x^{(\omega)}, X^* - X^{(\omega+1)} \right\rangle \\
&+ \ \left\langle R^{(\omega+1)}, \mathcal{M}_2(Y^* - Y^{(\omega+1)}) \right\rangle \\
&+ \ \frac{\|R^{(\omega+1)}\|^2}{\|R^{(\omega)}\|^2} \left\langle P_y^{(\omega)}, Y^* - Y^{(\omega+1)} \right\rangle \\
&= \ \left\langle R^{(\omega+1)}, R^{(\omega+1)} \right\rangle + \frac{\|R^{(\omega+1)}\|^2}{\|R^{(\omega)}\|^2} \left\{ \left\langle P_x^{(\omega)}, X^* - X^{(\omega)} \right\rangle \right. \\
&+ \ \left\langle P_y^{(\omega)}, Y^* - Y^{(\omega)} \right\rangle \\
&- \ \left. \frac{\|R^{(\omega)}\|^2}{\left\|P_x^{(\omega)}\right\|^2 + \left\|P_y^{(\omega)}\right\|^2} \left(\left\langle P_x^{(\omega)}, P_x^{(\omega)} \right\rangle + \left\langle P_y^{(\omega)}, P_y^{(\omega)} \right\rangle \right) \right\} \\
&= \ \left\|R^{(\omega+1)}\right\|^2 + \frac{\|R^{(\omega+1)}\|^2}{\|R^{(\omega)}\|^2} \left(\left\|R^{(\omega)}\right\|^2 - \left\|R^{(\omega)}\right\|^2 \right) = \left\|R^{(\omega+1)}\right\|^2.
\end{aligned}$$

By the principle of mathematical induction, the result (3.2) holds for $i = 1, 2, 3, \ldots$. □

Remark 3.4. *Lemma 3.3 implies that if there exists a positive integer k such that $P_x^{(k)} = 0$ and $P_y^{(k)} = 0$ but $R^{(k)} \neq 0$ then Problem 1 is not consistent.*

Let $\mathcal{K}$ be a matrix subspace consists of the matrices of the form $L = (L_1, L_2, \ldots, L_p)$ where $L_i \in \mathbb{C}^{r_i \times s_i}$ for $i = 1, 2, \ldots, p$. Assume that $m = r_1 s_1 + r_2 s_2 + \cdots + r_p s_p$, and m steps of Algorithm 1 have been performed. Therefore, $R^{(i)} \neq 0$ for $i = 1, 2, \ldots, m$. Lemma 3.2 shows that $R^{(i)}$, $i = 1, 2, \ldots, m$, is an orthogonal basis for $\mathcal{K}$. Now, we may conclude the following theorem.

Theorem 3.5. *Assume that Problem 1 is consistent. Suppose that $X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \ldots, X_q^{(1)})$ and $Y^{(1)} = (Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_q^{(1)})$ are arbitrary initial matrix groups where $X_j^{(1)} \in \mathbb{HC}_r^{n_j \times n_j}(P_j)$ and $Y_j^{(1)} \in \mathbb{SH}^{n_j \times n_j}$ for $j = 1, 2, \ldots, q$. Then, the solution groups of Problem 1 can be calculated by Algorithm 1 within at most $m + 1$ iteration steps, in the absence of roundoff errors, where $m = \sum_{i=1}^{p} r_i s_i$.*

Consider the system of matrix equations (2.1). As seen, Problem 1 is solvable if and only if (2.1) is consistent. Suppose that the matrices $P_j \in \mathbb{SOR}^{n_j \times n_j}$, $j \in I[1,q]$ are given. Let $R_{j1} = P_j$ and $R_{j2} = I_{n_j}$, where I_n is an $n \times n$ identity matrix.

We define the matrices $\psi_1^{(\ell)} = (\psi_{11}^{(\ell)}, \psi_{12}^{(\ell)}, \ldots, \psi_{1q}^{(\ell)})$ and $\psi_2^{(\ell)} = (\psi_{21}^{(\ell)}, \psi_{22}^{(\ell)}, \ldots, \psi_{2q}^{(\ell)})$, such that the jth columns of $\psi_1^{(\ell)}$ and $\psi_2^{(\ell)}$ have the following forms, respectively, ($\ell = 1, 2$, and $j = 1, 2, \ldots, q$)

$$\psi_{1j}^{(\ell)} = \begin{pmatrix} \sum_{\lambda=1}^{t_1^{(\ell)}} (B_{1j\lambda}^{(\ell)})^T \otimes A_{1j\lambda}^{(\ell)} + \sum_{\nu=1}^{t_3^{(\ell)}} ((F_{1j\nu}^{(\ell)})^T \otimes E_{1j\nu}^{(\ell)}) P_\ell(n_j, n_j) \\ \vdots \\ \sum_{\lambda=1}^{t_1^{(\ell)}} (B_{pj\lambda}^{(\ell)})^T \otimes A_{pj\lambda}^{(\ell)} + \sum_{\nu=1}^{t_3^{(\ell)}} ((F_{pj\nu}^{(\ell)})^T \otimes E_{pj\nu}^{(\ell)}) P_\ell(n_j, n_j) \\ (-1)^{\ell-1} \left(\sum_{\lambda=1}^{t_1^{(\ell)}} \overline{A_{1j\lambda}^{(\ell)}} \otimes (B_{1j\lambda}^{(\ell)})^H + \sum_{\nu=1}^{t_3^{(\ell)}} (\overline{E_{1j\nu}^{(\ell)}} \otimes (F_{1j\nu}^{(\ell)})^H) P_\ell(n_j, n_j) \right) \\ \vdots \\ (-1)^{\ell-1} \left(\sum_{\lambda=1}^{t_1^{(\ell)}} \overline{A_{pj\lambda}^{(\ell)}} \otimes (B_{pj\lambda}^{(\ell)})^H + \sum_{\nu=1}^{t_3^{(\ell)}} (\overline{E_{pj\nu}^{(\ell)}} \otimes (F_{pj\nu}^{(\ell)})^H) P_\ell(n_j, n_j) \right) \\ \sum_{\lambda=1}^{t_1^{(\ell)}} (B_{1j\lambda}^{(\ell)})^T R_{j\ell}^T \otimes A_{1j\lambda}^{(\ell)} R_{j\ell} + \sum_{\nu=1}^{t_3^{(\ell)}} ((F_{1j\nu}^{(\ell)})^T R_{j\ell} \otimes E_{1j\nu}^{(\ell)} R_{j\ell}^T) P_\ell(n_j, n_j) \\ \vdots \\ \sum_{\lambda=1}^{t_1^{(\ell)}} (B_{pj\lambda}^{(\ell)})^T R_{j\ell}^T \otimes A_{pj\lambda}^{(\ell)} R_{j\ell} + \sum_{\nu=1}^{t_3^{(\ell)}} ((F_{pj\nu}^{(\ell)})^T R_{j\ell} \otimes E_{pj\nu}^{(\ell)} R_{j\ell}^T) P_\ell(n_j, n_j) \\ (-1)^{\ell-1} \left(\sum_{\lambda=1}^{t_1^{(\ell)}} \overline{A_{1j\lambda}^{(\ell)}} R_{j\ell}^T \otimes B_{1j\lambda}^{(\ell)} R_{j\ell} + \sum_{\nu=1}^{t_3^{(\ell)}} (\overline{E_{1j\nu}^{(\ell)}} R_{j\ell} \otimes (F_{1j\nu}^{(\ell)})^H R_{j\ell}^T) P_\ell(n_j, n_j) \right) \\ \vdots \\ (-1)^{\ell-1} \left(\sum_{\lambda=1}^{t_1^{(\ell)}} \overline{A_{pj\lambda}^{(\ell)}} R_{j,\ell}^T \otimes B_{pj\lambda}^{(\ell)} R_{j\ell} + \sum_{\nu=1}^{t_3^{(\ell)}} (\overline{E_{pj\nu}^{(\ell)}} R_{j\ell} \otimes (F_{pj\nu}^{(\ell)})^H R_{j\ell}^T) P_\ell(n_j, n_j) \right) \end{pmatrix},$$

$$\psi_{2,j}^{(\ell)} = \begin{pmatrix} \sum_{\mu=1}^{t_2^{(\ell)}} (D_{1j\mu}^{(\ell)})^T \otimes C_{1j\mu}^{(\ell)} \\ \vdots \\ \sum_{\mu=1}^{t_2^{(\ell)}} (D_{pj\mu}^{(\ell)})^T \otimes C_{pj\mu}^{(\ell)} \\ (-1)^{\ell-1} \left(\sum_{\mu=1}^{t_2^{(\ell)}} \overline{C_{1j\mu}^{(\ell)}} \otimes D_{1j\mu}^{(\ell)} \right) \\ \vdots \\ (-1)^{\ell-1} \left(\sum_{\mu=1}^{t_2^{(\ell)}} \overline{C_{pj\mu}^{(\ell)}} \otimes D_{pj\mu}^{(\ell)} \right) \\ \sum_{\mu=1}^{t_2^{(\ell)}} (D_{1j\mu}^{(\ell)})^T R_{j\ell} \otimes C_{1j\gamma}^{(\ell)} \overline{R}_{j\ell} \\ \vdots \\ \sum_{\mu=1}^{t_2^{(\ell)}} (D_{pj\mu}^{(\ell)})^T R_{j\ell} \otimes C_{pj\gamma}^{(\ell)} \overline{R}_{j\ell} \\ (-1)^{\ell-1} \left(\sum_{\mu=1}^{t_2^{(\ell)}} \overline{C_{1j\mu}^{(\ell)}} R_{j\ell} \otimes (D_{1j\mu}^{(\ell)})^H \overline{R}_{j\ell} \right) \\ \vdots \\ (-1)^{\ell-1} \left(\sum_{\mu=1}^{t_2^{(\ell)}} \overline{C_{pj\mu}^{(\ell)}} R_{j\ell} \otimes (D_{pj\mu}^{(\ell)})^H \overline{R}_{j\ell} \right) \end{pmatrix}.$$

Suppose that $\Psi_1 = [\psi_1^{(1)}, \psi_2^{(1)}]$, $\Psi_2 = [\psi_1^{(2)}, \psi_2^{(2)}]$, $\widehat{X} = [\widehat{X}_1^T, \widehat{X}_2^T]^T$, $\widehat{Y} = [\widehat{Y}_1^T, \widehat{Y}_2^T]^T$ and $\widehat{L} = [\widehat{L}_1^T, \widehat{L}_2^T, \widehat{L}_1^T, \widehat{L}_2^T]^T$ where,

$$\widehat{X}_1 = \begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \\ \vdots \\ \mathrm{vec}(X_q) \end{pmatrix}, \quad \widehat{X}_2 = \begin{pmatrix} \mathrm{vec}(\overline{X}_1) \\ \mathrm{vec}(\overline{X}_2) \\ \vdots \\ \mathrm{vec}(\overline{X}_q) \end{pmatrix}, \quad \widehat{Y}_1 = \begin{pmatrix} \mathrm{vec}(Y_1) \\ \mathrm{vec}(Y_2) \\ \vdots \\ \mathrm{vec}(Y_q) \end{pmatrix}, \quad \widehat{Y}_2 = \begin{pmatrix} \mathrm{vec}(\overline{Y}_1) \\ \mathrm{vec}(\overline{Y}_2) \\ \vdots \\ \mathrm{vec}(\overline{Y}_q) \end{pmatrix},$$

$$\widehat{L}_1 = \begin{pmatrix} \mathrm{vec}(L_1) \\ \mathrm{vec}(L_2) \\ \vdots \\ vec(L_q) \end{pmatrix}, \qquad \text{and} \qquad \widehat{L}_2 = \begin{pmatrix} \mathrm{vec}(L_1^H) \\ \mathrm{vec}(L_2^H) \\ \vdots \\ \mathrm{vec}(L_q^H) \end{pmatrix},$$

It is not difficult to show that (2.1) is equivalent to the following linear system of equations:

$$\Psi_1 \widehat{X} + \Psi_2 \widehat{Y} = \widehat{L}.$$

For an arbitrary given matrix group $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_p)$, where $\Lambda_i \in \mathbb{C}^{r_i \times s_i}$ for $i \in I[1,p]$, we define the matrix group $\Phi_\ell = (\varphi_1^{(\ell)}, \varphi_2^{(\ell)}, \ldots, \varphi_q^{(\ell)})$, $\ell = 1, 2$, such that

$$\Phi_1 = 4\mathcal{D}_1(\Lambda), \quad \text{and} \quad \Phi_2 = 2\mathcal{D}_2(\Lambda).$$

Let us assume that

$$\widetilde{\Phi}_\ell^T = ((\mathrm{vec}(\varphi_1^{(\ell)}))^T, (\mathrm{vec}(\varphi_2^{(\ell)}))^T, \dots, (\mathrm{vec}(\varphi_q^{(\ell)}))^T, (\mathrm{vec}(\overline{\varphi_1^{(\ell)}}))^T, (\mathrm{vec}(\overline{\varphi_2^{(\ell)}}))^T, \dots, (\mathrm{vec}(\overline{\varphi_q^{(\ell)}}))^T), \quad \ell = 1, 2,$$

and $\widehat{\Lambda} = (\widehat{\Lambda}_1^T, \widehat{\Lambda}_2^T, \widehat{\Lambda}_1^T, \widehat{\Lambda}_2^T)^T$ where

$$\widehat{\Lambda}_1 = \begin{pmatrix} \mathrm{vec}(\Lambda_1) \\ \mathrm{vec}(\Lambda_2) \\ \vdots \\ vec(\Lambda_p) \end{pmatrix}, \quad \text{and} \quad \widehat{\Lambda}_2 = \begin{pmatrix} \mathrm{vec}(\Lambda_1^H) \\ \mathrm{vec}(\Lambda_2^H) \\ \vdots \\ \mathrm{vec}(\Lambda_p^H) \end{pmatrix}.$$

By some easy computations, it is deduced that

$$\begin{pmatrix} \widetilde{\Phi}_1 \\ \widetilde{\Phi}_2 \end{pmatrix} = \begin{pmatrix} \Psi_1^H \\ \Psi_2^H \end{pmatrix} \widehat{\Lambda}. \tag{3.3}$$

Or equivalently,

$$\begin{pmatrix} \widetilde{\Phi}_1 \\ \widetilde{\Phi}_2 \end{pmatrix} \in R\left(\begin{pmatrix} \Psi_1^H \\ \Psi_2^H \end{pmatrix}\right).$$

Therefore, if the initial matrix groups are chosen such that

$$X^{(1)} = 4\mathcal{D}_1(\Lambda) \quad \text{and} \quad Y^{(1)} = 2\mathcal{D}_2(\Lambda). \tag{3.4}$$

Then, the sequences of approximate matrix groups $X^{(j)}, Y^{(j)}$, produced by Algorithm 1, belong to $R\left([\Psi_1, \Psi_2]^H\right)$.

For a consistent linear system $Ax = b$, it is well-known that the solutions of the linear system $Ax = b$ are computed by $x = A^+b + N(A)$. The notation A^+ denotes the well-known Moore-Penrose pseudo inverse of the matrix A. Hence, $x^* = A^+b$ is the minimum norm solution. As $R(A^+) = R(A^H)$, we deduce that $x^* \in R(A^H)$ is a least 2-norm solution of the linear system $Ax = b$.

Theorem 3.6. *Suppose that Problem 1 is consistent. If $X^{(1)} = 4\mathcal{D}_1(\Lambda)$ and $Y^{(1)} = 2\mathcal{D}_2(\Lambda)$ where Λ is an arbitrary matrix group defined as before. Then the solution groups X^* and Y^*, generated by Algorithm 1, are the least-norm solution groups.*

3.2. **An approach for solving Problems 2.** Assume that the matrices $P_j \in \mathbb{SOC}^{n_j \times n_j}$ for $j \in I[1, q]$ are given. Suppose that Problem 1 is consistent, so its solution group set $S_{\mathcal{M}}$ is nonempty. Considering the matrix groups $\Gamma_x = (\Gamma_{1x}, \Gamma_{2x}, \dots, \Gamma_{qx})$ and $\Gamma_y = (\Gamma_{1y}, \Gamma_{2y}, \dots, \Gamma_{qy})$, where $\Gamma_{jx} \in \mathbb{HC}_r^{n_j \times n_j}(P_j)$ and $\Gamma_{jy} \in \mathbb{SH}^{n_j \times n_j}$ for $j \in I[1, q]$. It is not difficult to verify that the coupled linear matrix equations

$$\mathcal{M}(X, Y) = L,$$

is equivalent to the following coupled linear matrix equations

$$\mathcal{M}(V_x, V_y) = \tilde{L}, \tag{3.5}$$

where $\tilde{L} = L - \mathcal{M}(\Gamma_x, \Gamma_y)$, $V_{jx} = X_j - \Gamma_{jx}$ and $V_{jy} = Y_j - \Gamma_{jy}$ for $j \in I[1, q]$. In order to find the solution of Problem 2, employing Algorithm 1, we first find the least-norm solution groups $V_x^* = (V_{1x}^*, V_{2x}^*, \dots, V_{qx}^*)$ and $V_y^* = (V_{1y}^*, V_{2y}^*, \dots, V_{qy}^*)$ of (3.5). Then, the Hermitian reflexive and skew-Hermitian solution groups for Problem 2, respectively, can be computed by

$$\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2, \dots, \widetilde{X}_q), \quad \text{and} \quad \widetilde{Y} = (\widetilde{Y}_1, \widetilde{Y}_2, \dots, \widetilde{Y}_q),$$

where $\widetilde{X}_j = V_{jx}^* + \Gamma_{jx}$ and $\widetilde{Y}_j = V_{jy}^* + \Gamma_{jy}$ for $j \in I[1, q]$.

4. Numerical experiments

In this section, we present two numerical examples to show the effectiveness of the proposed algorithm. All the numerical experiments presented in this section have been computed in double precision with some Matlab codes on a Pentium 4 PC, with a 3.06 GHz CPU and 1.00GB of RAM.

Example 4.1. We consider the following matrix equation

$$
\begin{aligned}
(4.1)\qquad & A_{111}^{(1)}X_1B_{111}^{(1)} + A_{112}^{(1)}X_1B_{112}^{(1)} + C_{111}^{(1)}\overline{X}_1D_{111}^{(1)} + E_{111}^{(1)}X_1^TF_{111}^{(1)} \\
& +A_{111}^{(2)}Y_1B_{111}^{(2)} + C_{111}^{(2)}\overline{Y}_1D_{111}^{(2)} + C_{112}^{(2)}\overline{Y}_1D_{112}^{(2)} + E_{111}^{(2)}Y_1^TF_{111}^{(2)} = L_1,
\end{aligned}
$$

where

$$
A_{111}^{(1)} = \begin{pmatrix} 4 & 2+i & 1 \\ 2 & 5 & 1 \\ 1 & 2 & 1+4i \end{pmatrix},\ B_{111}^{(1)} = \begin{pmatrix} i & 2-i & 1 \\ 1 & 2 & 0 \\ 1+i & -i & 2 \end{pmatrix},\ A_{112}^{(1)} = \begin{pmatrix} 1 & 2 & i \\ 1+i & 2-i & i \\ -1 & i & -2 \end{pmatrix},
$$

$$
B_{112}^{(1)} = \begin{pmatrix} 2 & 1 & 2 \\ 1-i & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix},\ C_{111}^{(1)} = \begin{pmatrix} 1 & 2 & -1+2i \\ -2+i & -1 & 1 \\ 1 & 2-i & 1+i \end{pmatrix},
$$

$$
D_{111}^{(1)} = \begin{pmatrix} 1 & 2 & -1+i \\ -1 & 2 & 1 \\ 2+i & 1-2i & 1-i \end{pmatrix},\ E_{111}^{(1)} = \begin{pmatrix} 1+i & 1-i & 1 \\ -2i & 2 & 3 \\ 1-i & 1 & 1-i \end{pmatrix},
$$

$$
F_{111}^{(1)} = \begin{pmatrix} 1 & 2 & 2-2i \\ i & 1+i & 2 \\ i & 1+2i & -2 \end{pmatrix},\ A_{111}^{(2)} = \begin{pmatrix} 1-i & 2 & 1+2i \\ 2 & -1 & 1 \\ 1 & 2 & i \end{pmatrix},\ B_{111}^{(2)} = \begin{pmatrix} i & 2 & 1 \\ 1 & 2 & 1 \\ i & -i & 1 \end{pmatrix},
$$

$$
C_{111}^{(2)} = \begin{pmatrix} 2 & 2 & i \\ 1 & 2i & i \\ -1 & i & 1+2i \end{pmatrix},\ D_{111}^{(2)} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix},\ C_{112}^{(2)} = \begin{pmatrix} i & 2 & -1 \\ -2+i & -1 & 1 \\ 1 & 2 & i \end{pmatrix},
$$

$$
D_{112}^{(2)} = \begin{pmatrix} 2i & 2 & -1 \\ -1 & 2 & 1 \\ 2 & 1 & i \end{pmatrix},\ E_{111}^{(2)} = \begin{pmatrix} i & 1-2i & i \\ 2i & 2 & 1 \\ i & 1 & 1+i \end{pmatrix},\ F_{111}^{(2)} = \begin{pmatrix} i & 2 & -i \\ 2+i & 1-3i & 2-4i \\ i & 1-2i & 1 \end{pmatrix},
$$

and

$$
L_1 = \begin{pmatrix} 59+62i & 236-42i & 112-77i \\ 6-13i & 158+61i & 95-9i \\ 20+100i & 107+13i & 35+82i \end{pmatrix},
$$

in which $i = \sqrt{-1}$. The exact solution of (4.1) is given by (X_1^*, Y_1^*), where

$$
X_1^* = \begin{pmatrix} 2 & 4 & 1-i \\ 4 & 2 & -1+i \\ 1+i & -1-i & 2 \end{pmatrix},\quad \text{and}\quad \mathrm{Y}_1^* = \begin{pmatrix} 6i & 3i & -1-4i \\ 3i & 6i & 1-2i \\ 1-4i & -1-2i & 2i \end{pmatrix}.
$$

We mention that $X_1^* \in \mathbb{HC}_r^{3\times 3}(P_1)$ where

$$
P_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{SOC}^{3\times 3},
$$

and $Y_1^* \in \mathbb{SH}^{3\times 3}$.

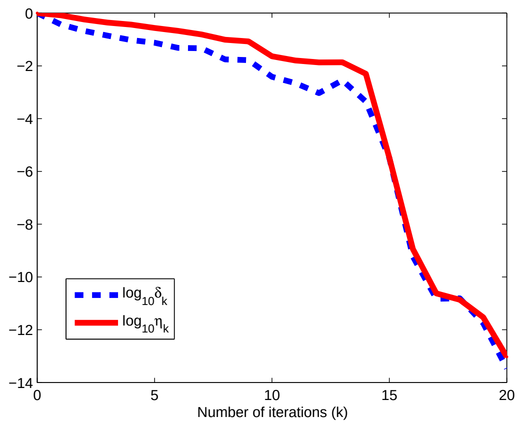

FIGURE 1. $\log_{10}\delta_k$ and $\log_{10}\eta_k$ versus k to solve Problem 1 corresponding system (4.1).

.

We apply Algorithm 1 to solve Problem 1 corresponding to system (4.1). The initial guess was taken to be the zero matrix $(X_1^{(0)}, Y_1^{(0)}) = (0, 0)$ and the stopping criterion

$$\delta_k = \frac{\|R^{(k)}\|_F}{\|R^{(0)}\|_F} < 10^{-12},$$

was used, where $R^{(k)}$ is the residual of (4.1) at kth iteration. In this case, the method converges in 20 iterations. For more details, $\log_{10}\delta_k$ together with $\log_{10}\eta_k$ are depicted in FIGURE 1, where

$$\eta_k = \max\{\frac{\|X_1^* - X_1^{(k)}\|_F}{\|X_1^*\|_F}, \frac{\|Y_1^* - Y_1^{(k)}\|_F}{\|Y_1^*\|_F}\}.$$

Now, let

$$\Gamma_{1x} = \begin{pmatrix} 5 & 2 & 1-2i \\ 2 & 5 & -1+2i \\ 1+2i & -1-2i & 2 \end{pmatrix}, \quad \text{and} \quad \Gamma_{1y} = \begin{pmatrix} 2i & i & 2-i \\ i & i & 1-i \\ -2-i & -1-i & -i \end{pmatrix}.$$

If we apply Algorithm 1 to solve Problem 2 corresponding to system (4.1), then the method converges in 18 iterations. Convergence history of the method is displayed in FIGURE 2. All of the assumptions are as before.

Example 4.2. In this example, we consider the system of matrix equations

$$\text{(4.2)} \qquad \begin{cases} A_{111}^{(1)} X_1 B_{111}^{(1)} + C_{111}^{(1)} \overline{X}_1 D_{111}^{(1)} + E_{121}^{(1)} X_2^T F_{121}^{(1)} \\ \qquad + A_{111}^{(2)} Y_1 B_{111}^{(2)} + C_{121}^{(2)} \overline{Y}_2 D_{121}^{(2)} + E_{121}^{(2)} Y_2^T F_{121}^{(2)} = L_1, \\ A_{211}^{(1)} X_1 B_{211}^{(1)} + C_{211}^{(1)} \overline{X}_1 D_{211}^{(1)} + E_{221}^{(1)} X_2^T F_{221}^{(1)} \\ \qquad + A_{211}^{(2)} Y_1 B_{211}^{(2)} + C_{221}^{(2)} \overline{Y}_2 D_{221}^{(2)} + E_{221}^{(2)} Y_2^T F_{221}^{(2)} = L_2, \end{cases}$$

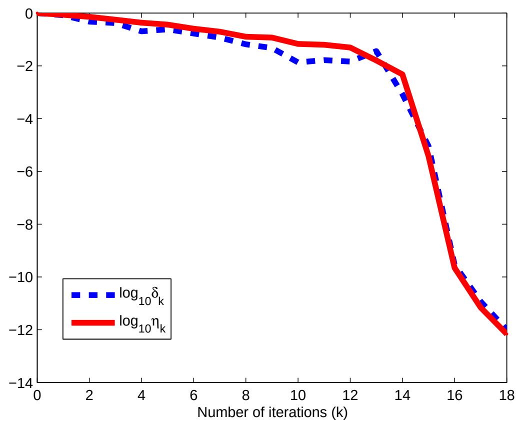

FIGURE 2. $\log_{10}\delta_k$ and $\log_{10}\eta_k$ versus k to solve Problem 2 corresponding system (4.1).

.

where

$$A_{111}^{(1)}=\begin{pmatrix}1 & 1+2i\\ 2 & 4+i\end{pmatrix},\ B_{111}^{(1)}=\begin{pmatrix}i & i\\ 1 & 1-i\end{pmatrix},\ C_{111}^{(1)}=\begin{pmatrix}2 & i\\ 2i & -2\end{pmatrix},\ D_{111}^{(1)}=\begin{pmatrix}-2i & 1\\ i & 1+i\end{pmatrix},$$

$$E_{121}^{(1)}=\begin{pmatrix}1-i & 1+i\\ i & -2\end{pmatrix},\ F_{121}^{(1)}=\begin{pmatrix}3 & 2i\\ 1-i & i\end{pmatrix},\ A_{111}^{(2)}=\begin{pmatrix}i & 2i\\ 5 & -2\end{pmatrix},\ B_{111}^{(2)}=\begin{pmatrix}2i & 2\\ 3i & -i\end{pmatrix},$$

$$C_{121}^{(2)}=\begin{pmatrix}1 & 1+i\\ -2 & 2i\end{pmatrix},\ D_{121}^{(2)}=\begin{pmatrix}2 & -2\\ -3 & 1\end{pmatrix},\ E_{121}^{(2)}=\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix},\ F_{121}^{(2)}=\begin{pmatrix}-2 & 2\\ -3 & -i\end{pmatrix},$$

$$A_{211}^{(1)}=\begin{pmatrix}1-i & i\\ 1 & -i\end{pmatrix},\ B_{211}^{(1)}=\begin{pmatrix}i & 1+2i\\ 1 & i\end{pmatrix},\ C_{211}^{(1)}=\begin{pmatrix}1 & 2i\\ -2i & -1\end{pmatrix},$$

$$D_{211}^{(1)}=\begin{pmatrix}2 & 1\\ 1-i & 1-i\end{pmatrix},\ E_{221}^{(1)}=\begin{pmatrix}1+i & 1-i\\ 2i & -3\end{pmatrix},\ F_{221}^{(1)}=\begin{pmatrix}1 & 2i\\ 1+i & 1-i\end{pmatrix},$$

$$A_{211}^{(2)}=\begin{pmatrix}i & 2-i\\ 1+i & -2i\end{pmatrix},\ B_{211}^{(2)}=\begin{pmatrix}i & 1\\ 1-2i & -i\end{pmatrix},\ C_{221}^{(2)}=\begin{pmatrix}2 & 1-i\\ -2 & 1+2i\end{pmatrix},$$

$$D_{221}^{(2)}=\begin{pmatrix}1+i & -2i\\ -i & 1+i\end{pmatrix},\ E_{221}^{(2)}=\begin{pmatrix}0 & -i\\ 0 & 1\end{pmatrix},\ F_{221}^{(2)}=\begin{pmatrix}1 & -i\\ 2i & i\end{pmatrix},$$

and

$$L_1=\begin{pmatrix}-139+181i & -89-35i\\ -44-125i & 99-32i\end{pmatrix},\quad L_2=\begin{pmatrix}-37-16i & 215-3i\\ -66-38i & 327-149i\end{pmatrix}.$$

The exact solution of Eq. (4.2) is the matrix groups (X_1^*, X_2^*) and (Y_1^*, Y_2^*) where

$$X_1^*=\begin{pmatrix}4 & 2\\ 2 & 4\end{pmatrix},\quad X_2^*=\begin{pmatrix}-54 & 22\\ 22 & -21\end{pmatrix},$$
$$Y_1^*=\begin{pmatrix}2i & 1-i\\ -1-i & 2i\end{pmatrix},\quad Y_2^*=\begin{pmatrix}i & 1-2i\\ -1-2i & i\end{pmatrix}.$$

We mention that $X_i^*\in\mathbb{HC}_r^{2\times2}(P_i)$, $i=1,2$, where

$$P_1=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\in\mathbb{SOC}^{2\times2},\quad P_2=\frac{1}{5}\begin{pmatrix}3 & -4\\ -4 & -3\end{pmatrix}\in\mathbb{SOC}^{2\times2},$$

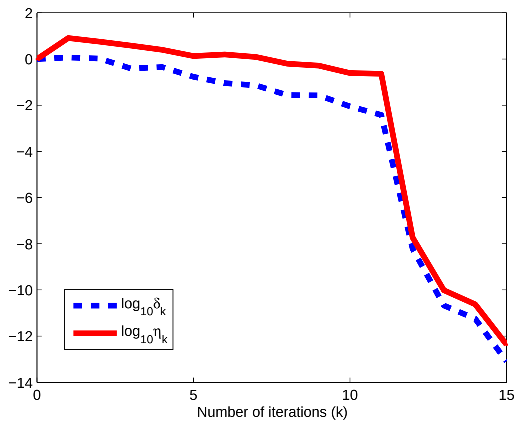

FIGURE 3. $\log_{10}\delta_k$ and $\log_{10}\eta_k$ versus k to solve Problem 1 corresponding system (4.2).

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and $Y_1^*, Y_2^* \in \mathbb{SH}^{2\times 2}$. To solve Problem 1 corresponding to system (4.2), we use Algorithm 1. The initial guess was taken to be the zero matrix $(X_1^{(0)}, Y_1^{(0)}) = (0,0)$ and the stopping criterion

$$\delta_k = \max\{\frac{\|R_i^{(k)}\|_F}{\|R_i^{(0)}\|_F} : i = 1,2\} < 10^{-12},$$

was utilized where $R_i^{(k)}$ is the ith residual of (4.2) at kth iteration. In this case, the method converges in 15 iterations. In FIGURE 1, $\log_{10}\delta_k$ together with $\log_{10}\eta_k$ where

$$\eta_k = \max\{\frac{\|X_i^* - X_i^{(k)}\|_F}{\|X_i^*\|_F}, \frac{\|Y_i^* - Y_i^{(k)}\|_F}{\|Y_i^*\|_F} : i = 1,2\},$$

are plotted.

Now, let

$$\Gamma_{1x} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \Gamma_{2x} = \begin{pmatrix} 31 & -8 \\ -8 & 19 \end{pmatrix}.$$

$$\Gamma_{1y} = \begin{pmatrix} i & 1+i \\ -1+i & i \end{pmatrix}, \quad \Gamma_{2y} = \begin{pmatrix} 2i & i \\ i & i \end{pmatrix}.$$

If we apply Algorithm 1 to solve Problem 2 corresponding to system (4.2), then the method converges in 14 iterations. Convergence history of the method is illustrated in FIGURE 2. All of the assumptions are as before.

5. CONCLUSION

We have considered a general class of coupled linear matrix equations over the complex number field. The unknown matrix groups X and Y appear in the mentioned coupled linear matrix equations. The purpose of this paper is to find the solution of two main problems. The first problem, (Problem 1), concerned with finding the matrix groups X, Y such that X and Y are the groups of Hermitian reflexive and skew-Hermitian matrices, respectively. In the case that the first problem is consistent, the solution of the second problem (Problem 2) is the optimal approximate Hermitian reflexive and skew-Hermitian solution groups to the given arbitrary matrix groups Γ_x and Γ_y. An iterative algorithm has been proposed to solve Problem 1. The solvability of Problem 1 can be automatically determined by the algorithm. When Problem 1 is

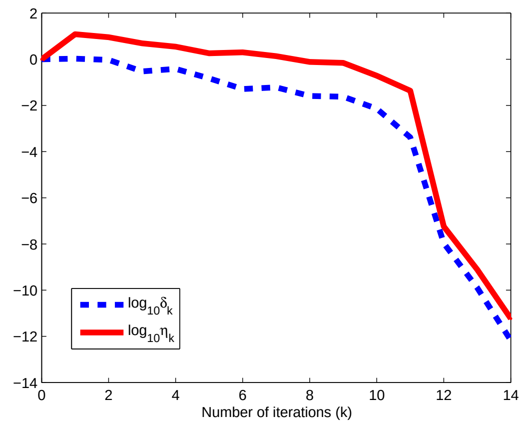

FIGURE 4. $\log_{10}\delta_k$ and $\log_{10}\eta_k$ versus k to solve Problem 2 corresponding system (4.2).
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consistent, we have proved that in absence of round of errors the algorithm is convergent within finite number of iterations. In addition, an approach has been given for solving the second problem. In order to illustrate the feasibly of the proposed algorithm for solving Problems 1 and 2, some numerical results have been presented.

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