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An Iterative Method to Compute Minimum norm Solutions of Ill-posed Problems in Hilbert Spaces

Meisam Jozi · Saeed Karimi · Davod Khojasteh Salkuyeh

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Abstract We study an algorithm to compute minimum norm solution of ill-posed problems in Hilbert spaces and investigate its regularizing properties with discrepancy principle stopping rule. This algorithm results from straightly applying the LSQR method to the main problem before discretizing. In fact, the proposed algorithm obtains a sequence of approximate solutions of the original problem. In order to test the new algorithm, it is implemented to solve system of linear integral equations of the first kind and some examples are given. Moreover, we compare the presented algorithm with the Tikhonov regularization method to compute the least norm solution when there are more than one solution.

Keywords Ill-posed problem · first kind equations · regularization method · $\mathcal{LS}$-algorithm · minimum norm

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1 Introduction

Let $\mathbb{A} : \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator from a Hilbert space $\mathbb{X}$ into a Hilbert space $\mathbb{Y}$. For a given $y \in$ range $\mathbb{A}(\mathbb{X})$ of $\mathbb{A}$, we consider the linear equation

$$\mathbb{A}x = y. \tag{1}$$

This equation is ill-posed if $\mathbb{A}$ is not boundedly invertible [1,19,20]. This makes the difficult to solve by straightforward application of numerical methods, developed to solve well-posed problems. A broad class of so-called inverse problems that arise in physics, technology and other branches of science, belongs to the class of ill-posed problems. The importance of ill-posed problems causes many researchers try to solve such problems [2,6,7,11,26,29]. A general strategy for solving ill-posed problem (1) is the regularization technique [19,20,26]. So far, many regularization schemes have been proposed including the Tikhonov's method, the Landweber's iterations and so on [19]. Regularization methods depend on a parameter called regularization parameter. The choice of this parameter is crucial to obtain a good solution. Since there is some inherent noise in the data y, in the discussion of regularization methods, Eq. (1) is considered with a perturbed right hand side $y = y^{exact} + e$, where e and y^{exact} denote the noise and the unknown error-free right-hand side, respectively. Then the regularization parameter α depending on error level $\|e\| \approx \delta$ is chosen in order to achieve an acceptable total error for the regularized solution. The strategies for choosing the regularization parameter are usually classified into two categories: a priori and a posteriori.

Meisam Jozi
Department of Mathematics, Persian Gulf University, Bushehr, Iran
E-mail: maisam.j63@gmail.com
Saeed Karimi
Department of Mathematics, Persian Gulf University, Bushehr, Iran
E-mail: karimi@pgu.ac.ir, karimijafarbigloo@gmail.com

Davod Khojasteh Salkuyeh
Department of Mathematics, University of Guilan, Rasht, Gilan
E-mail: khojasteh@guilan.ac.ir

An a priori choice depends on some information on smoothness properties of the exact solution which are not known in advance in applied problems. Hence, a posteriori choice, choosing during the algorithm, is more applicable. A well known a posteriori method suggested by Morozov, is discrepancy or residue principle [7,19,20,22]. His idea is based on the fact that the magnitude of residual should not be smaller than the magnitude of the noise in the right hand side. Premised on the idea behind the discrepancy principle, iterative methods have an inherent regularizing property when applied directly to (1) [4,5,13, 17,19]. That is, the iterations is stopped as soon as the residual $\|\mathbb{A}x_k - y\| \leq \ell\delta$ for some $\ell > 1$, where x_k is the kth approximate solution. In other words, the number of iterations plays the role of the regularization parameter which is controlled by an suitable stopping rule.

Moreover, it is frequently interesting to find the solution which has minimum norm in $\mathbb{X}$, if there are more than one solution of (1) [17,21,23]. Several methods have been presented in this regard such as the Tikhonov method [26,27], collocation method [12,19,20] and the Marti's method [21].

Recently, the authors proposed a least squares based algorithm, namely $\mathcal{LS}$-algorithm, to solve linear Fredholm integral equations [18] which is based on the LSQR method, for solving linear systems of equations [24]. The $\mathcal{LS}$-algorithm generates a sequence of orthonormal functions and uses them as a basis functions to approximate solution of linear integral equations.

In this paper, we implement the LSQR algorithm to the main problem ($\mathcal{LSAM}$) for computing the minimum norm solution of ill-posed problem (1) and compare it with the Tikhonov's method [26,27]. The $\mathcal{LSAM}$ algorithm generates a sequence of approximate solutions which minimizes the residual norm $\|\mathbb{A}x - y\|$ with respect to the norm produced by the inner product of space $\mathbb{Y}$, where x may range over a subspace of finite dimension. Also, we study the regularizing properties of $\mathcal{LSAM}$ by using the discrepancy principle in context of iterative methods. In order to test our method, we implement it to the system of first kind integral equations.

A common way almost in all literature is to reduce the ill-posed problem to an associated discrete algebraic ill-conditioned system and then the regularization techniques are implemented to the obtained algebraic system. For instance in [2–5,13,16,19,25], the LSQR method is considered as a regularization technique to solve a linear system arising from discretization of the first kind integral equations with smooth kernel. It is worth noting that the LSQR algorithm is analytically equivalent to the CGLS algorithm but has more favorable numerical and stability properties [10,15,24]. Therefore, $\mathcal{LSAM}$ is numerically more stable than the CGLS algorithm for Eq. (1) in arbitrary Hilbert space. In the present work, two strategies are investigated: The first one is $\mathcal{LSAM}$ to solve ill-posed problem straightforwardly and the latter is the LSQR algorithm to solve associated algebraic ill-condition system. A major difference between these strategies is that $\mathcal{LSAM}$ obtains an approximate solution in finite dimensional space while the LSQR method gives solution in some grid points. The numerical results show that the solution obtained from the second strategy may be very far from the exact solution, however $\mathcal{LSAM}$ gives a very good approximation of the original solution. Also the regularization properties of $\mathcal{LSAM}$ are superior and more satisfactory than of the LSQR method.

The remainder of the paper is organized as follows. In Section 2, the $\mathcal{LSAM}$ method is described to compute the minimum norm solution in an abstract setting and its properties are discussed. The convergence and regularizing properties of $\mathcal{LSAM}$ are studied in Section 3. Section 4 is devoted to introducing the discretized $\mathcal{LSAM}$ and applying it to solve the system of linear integral equations of the first kind. The results of numerical experiments on a few examples are discussed in Section 5. Finally in Section 6 a brief conclusion is drawn.

2 Description of the method

In this section, we give an abstract framework of $\mathcal{LSAM}$ to compute the minimum norm solution of an ill-posed problem. Throughout the paper, we use the same symbols $\langle .,. \rangle$ and $\|.\|$ for the inner products and their corresponding norms on the Hilbert spaces $\mathbb{X}$ and $\mathbb{Y}$. Also, we recall that the adjoint of the operator $\mathbb{A}$ is $\mathbb{A}^* : \mathbb{Y} \longrightarrow \mathbb{X}$ satisfying

$$\langle \mathbb{A}u, v \rangle = \langle u, \mathbb{A}^* v \rangle,$$

for all $u \in \mathbb{X}$ and $v \in \mathbb{Y}$. As [18], we first present the bidiagonalization process, namely the $\mathcal{L}$-Bidiag process, for the linear operator $\mathbb{A}$. This process generates two sets of functions namely $v_1, v_2, \ldots \in \mathbb{X}$ and $u_1, u_2, \ldots \in \mathbb{Y}$ such that

$$\langle v_i, v_j \rangle = \delta_{ij} \qquad \text{and} \qquad \langle u_i, u_j \rangle = \delta_{ij}, \qquad i, j = 1, 2, \ldots,$$

where δ_{ij} is

$$\delta_{ij} = \begin{cases} 1, \, i = j, \\ 0, \, i \neq j. \end{cases}$$

The $\mathcal{L}$-bidiag process can be described as follows.

$\mathcal{L}$-Bidiag process:

$$\left.\begin{array}{c} \beta_1 u_1 = y, \quad \alpha_1 v_1 = \mathbb{A}^* u_1, \\ \beta_{i+1} u_{i+1} = \mathbb{A} v_i - \alpha_i u_i \\ \alpha_{i+1} v_{i+1} = \mathbb{A}^* u_{i+1} - \beta_{i+1} v_i \end{array}\right\}, \quad i = 1, 2, \ldots, \tag{2}$$

where $u_i \in \mathbb{X}$, $v_i \in \mathbb{Y}$ and the scalars $\alpha_i \geq 0$ and $\beta_i \geq 0$ are chosen such that $\|u_i\| = \|v_i\| = 1$. The following definitions are needed in the sequel.

Definition 1 Assume that $\mathbb{X}$ is a Hilbert space. Let $\lambda \in \mathbb{R}^k$, $G \in \mathbb{R}^{k \times k}$ and $V = [v_1, v_2, \ldots, v_k]$, where $v_i \in \mathbb{X}$, $i = 1, 2, ..., k$. Then

$$V \circ \lambda := \sum_{i=1}^{k} \lambda_i v_i,$$

and

$$V \circ G := [V \circ G(:, 1), V \circ G(:, 2), \ldots, V \circ G(:, k)],$$

where $G(:, j)$ denotes the jth column of G.

Definition 2 Let $\mathbb{X}$ and $\mathbb{Y}$ be two Hilbert spaces and $\mathbb{A} : \mathbb{X} \to \mathbb{Y}$ be a linear operator. Also $V = [v_1, v_2, \ldots, v_k]$, where $v_i \in \mathbb{X}$, $i = 1, \ldots, k$, then

$$\mathbb{A}V := [\mathbb{A}v_1, \mathbb{A}v_2, \ldots, \mathbb{A}v_k].$$

Now with the definitions

$$\mathbb{U}_k = [u_1, u_2, \ldots, u_k], \quad \mathbb{V}_k = [v_1, v_2, \ldots, v_k],$$

$$\mathbb{G}_k = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_k & \alpha_k \\ & & & \beta_{k+1} \end{pmatrix},$$

and by using Definitions 1 and 2, the recurrence formula (2) can be rewritten as

$$\begin{array}{l} \mathbb{U}_{k+1} \circ (\beta_1 e_1) = y, \\ \mathbb{A}\mathbb{V}_k = \mathbb{U}_{k+1} \circ \mathbb{G}_k, \\ \mathbb{A}^* \mathbb{U}_{k+1} = \mathbb{V}_k \circ \mathbb{G}_k^T + \alpha_{k+1} v_{k+1} e_{k+1}^T, \end{array} \tag{3}$$

where $\mathbb{G}^T$ denotes the transpose of $\mathbb{G}$ and e_i is the ith standard unit vector.

We now compute the approximate minimum norm solution of (1) by solving the following problem

$$\inf_x \|x\| \quad \text{subject to} \quad \mathbb{A}^* \mathbb{A} x = \mathbb{A}^* y, \tag{4}$$

where $\mathbb{A}^* \mathbb{A} x = \mathbb{A}^* y$ is equivalent to

$$\inf_x \|\mathbb{A} x - y\|. \tag{5}$$

The $\mathcal{LSAM}$ method forms solution estimations $x_k = \mathbb{V}_k \circ \Lambda_k$ for some $\Lambda_k \in \mathbb{R}^k$ at each iteration to minimize $\|r_k\|$, where $r_k = \mathbb{A} x_k - y$. By using (2) and (3) and that

$$(\mathbb{U}_{k+1} \circ \mathbb{G}_k) \circ \Lambda_k = \mathbb{U}_{k+1} \circ (\mathbb{G}_k \Lambda_k), \tag{6}$$

we have

$$r_k = y - \mathbb{A}(\mathbb{V}_k \circ \Lambda_k) = \beta_1 u_1 - \mathbb{U}_{k+1} \circ (\mathbb{G}_k \Lambda_k) = \mathbb{U}_{k+1} \circ (\beta_1 e_1 - \mathbb{G}_k \Lambda_k).$$

Since $\mathbb{U}_{k+1}$ is theoretically orthonormal, $\|r_k\| = \|\beta_1 e_1 - \mathbb{G}_k \Lambda_k\|_2$ and then the subproblem

$$\min_{\Lambda_k} \|\beta_1 e_1 - \mathbb{G}_k \Lambda_k\|_2,$$

is obtained, where $\|.\|_2$ is the Euclidean norm of $\mathbb{R}^k$. This is a least-squares problem with bidiagonal coefficient matrix which can be solved by the QR factorization using plane rotations. Therefore the $\mathcal{LSAM}$ method can be summarized as follows.

Algorithm 1 $\mathcal{LSAM}$
(Initialize.)
1. Set $x_0 = 0$ as a zero element
2. $\beta_1 = \|y\|, \quad u_1 = \frac{y}{\beta_1}, \quad \alpha_1 = \|\mathbb{A}^* u_1\|, \quad v_1 = \frac{\mathbb{A}^* u_1}{\alpha_1}$
3. Set $\omega_1 = v_1$, $\overline{\mu}_1 = \beta_1$, $\overline{\tau}_1 = \alpha_1$
4. For $i = 1, 2, \ldots$ until convergence, Do

(Continue the bidigonalization.)

5. $\chi_i = \mathbb{A} v_i - \alpha_i u_i$
6. $\beta_{i+1} = \|\chi_i\|$
7. $u_{i+1} = \frac{\chi_i}{\beta_{i+1}}$
8. $\varpi_i = \mathbb{A}^* u_{i+1} - \beta_{i+1} v_i$
9. $\alpha_{i+1} = \|\varpi_i\|$
10. $v_{i+1} = \frac{\varpi_i}{\alpha_{i+1}}$

(Construct and apply next orthogonal transformation.)

11. $\tau_i = \sqrt{\overline{\tau}_i^2 + \beta_{i+1}^2}$
12. $c_i = \frac{\overline{\tau}_i}{\tau_i}$
13. $s_i = \frac{\beta_{i+1}}{\tau_i}$
14. $\eta_{i+1} = s_i \alpha_{i+1}$
15. $\overline{\tau}_{i+1} = -c_i \alpha_{i+1}$
16. $\mu_i = c_i \overline{\mu}_i$
17. $\overline{\mu}_{i+1} = s_i \overline{\mu}_i$

(Update x and ω.)

18. $x_i = x_{i-1} + \frac{\mu_i}{\tau_i} \omega_i$
19. $\omega_{i+1} = v_{i+1} - \frac{\eta_{i+1}}{\tau_i} \omega_i$

(Test for convergence.)

20. If $|\overline{\mu}_{i+1}|$ is small enough then stop
21. EndDo

Proposition 1 *Let x_k be the approximate solution of (1), obtained by $\mathcal{LSAM}$, and let $r_k = b - \mathbb{A}x_k$ be its corresponding residual. If $\beta_k = 0$, for some k in $\mathcal{LSAM}$, then $r_k = 0$.*
Proof: According to Proposition 3.10 of [18] and step 13 of the $\mathcal{LS}$-algorithm, the proof is clear. □

Remark 1 It is easy to show that $\mathcal{LSAM}$ may breakdown when $\alpha_k = 0$, for some k.

Remark 2 In the ill-posed problem, unlike well-posed problem, the solution error may not tend to zero when the residual error tends to zero.

Now, to reach our goal, i.e., solution of the problem (4), we prove that $\mathcal{LSAM}$ returns an approximate solution to the minimum norm solution of Eq. (1). The following lemma is well-known and its proof is clear.

Lemma 1 *If $\mathbb{A} : \mathbb{X} \to \mathbb{Y}$ is a bounded linear operator and $g \in \mathbb{X}$ satisfies $\mathbb{A}^* \mathbb{A} g = 0$, then $g \in Null(\mathbb{A})$, where $Null(\mathbb{A})$ is the null space of $\mathbb{A}$.*

Theorem 2 *The $\mathcal{LSAM}$ method gives an approximate solution to the minimum norm solution.*
Proof: Assume that $\mathcal{LSAM}$ converges to $x^{\mathcal{L}}$, a solution of $\mathbb{A}x = y$, and let $\hat{x}$ be any other solution. Both of these solutions satisfy the normal equations $\mathbb{A}^*\mathbb{A}x = \mathbb{A}^*y$. Putting $e = \hat{x} - x^{\mathcal{L}}$, the difference between two normal equations $\mathbb{A}^*\mathbb{A}\hat{x} = \mathbb{A}^*y$ and $\mathbb{A}^*\mathbb{A}x^{\mathcal{L}} = \mathbb{A}^*y$ results in $\mathbb{A}^*\mathbb{A}e = 0$ and by Lemma 1 $\mathbb{A}e = 0$. On the other hand, from $\alpha_1 v_1 = \mathbb{A}^* u_1$ and $\alpha_{k+1} v_{k+1} = \mathbb{A}^* u_{k+1} - \beta_{k+1} v_k$ in (2), we have $v_1, v_2, ... \in \mathbb{A}^*(\mathbb{X})$. Therefore, by Theorem 15.8 in [20], $e \perp \text{span}\{v_1, v_2, ...\}$, hence

$$\begin{aligned} \|\hat{x}\|^2 - \|x^{\mathcal{L}}\|^2 = \ & \|x^{\mathcal{L}} + e\|^2 - \|x^{\mathcal{L}}\|^2 = \langle e, e\rangle + 2\langle x^{\mathcal{L}}, e\rangle \\ & = \|e\|^2 + 2\langle \sum_{j=1}^{\infty} \lambda_j v_j, e\rangle = \|e\|^2 \geq 0. \qquad \square \end{aligned}$$

The $\mathcal{LSAM}$ method can be applied to solve every bounded linear operator equation provided the adjoint of corresponding operator is accessible. This algorithm is interesting for several reasons.
(i) It is an iterative method and we do not need to store the basis functions produced during the algorithm

and that approximate solutions and residuals $\|\mathbb{A}x_k - y\|$ are easily computed and updated with simple recurrences. For these reasons, we have a fast algorithm.
(ii) The $\mathcal{LSAM}$ method gives an approximation of the minimum norm solution which is interesting when there are more than one solutions of Eq. (1) [8,17,21].
(iii) The $\mathcal{LSAM}$ method can be considered as a regularization strategy in the context of iteration methods which is the subject of the next section.

3 Convergence and regularizing properties

In this section, we study the convergence and regularizing properties of the proposed algorithm. As previously mentioned, $\mathcal{LSAM}$ results from formulating the LSQR method for operator equations. On the other hand, LSQR is mathematically equivalent to the conjugate gradient (CG) algorithm for least squares problems, applied to normal equations system. Thus, we discuss in an analogous fashion to that of the CG algorithm for normal equation

$$\mathbb{A}^*\mathbb{A}x = \mathbb{A}^*y. \tag{7}$$

The basic convergence properties of this algorithm have been studied in [19]. Therefore, after a brief explanation, we state the main result from [19] without giving proof.

Let $\mathbb{A} : \mathbb{X} \to \mathbb{Y}$ be a compact, linear and injective operator and $V_k = \text{span}\{v_1, ..., v_k\}$, be the orthonormal set produced by $\mathcal{L}$-Bidiag process. This set is the same as the set of directions $p^j,\ j = 0, 1, \ldots,$ in the algorithm CG formulated in [19, P. 56]. By Definition 3.6 and Proposition 3.8 in [18], we know that

$$V_k = \text{span}\{v_1, \mathbb{A}^*\mathbb{A}v_1, (\mathbb{A}^*\mathbb{A})^2v_1, ..., (\mathbb{A}^*\mathbb{A})^{k-1}v_1\},$$

and x_k minimizes the residual $\| \mathbb{A}x - y \|$ on V_k. Therefore x_k can be written as

$$x_k = P_{k-1}(\mathbb{A}^*\mathbb{A})v_1 = \mathcal{P}_{k-1}(\mathbb{A}^*\mathbb{A})\mathbb{A}^*y, \tag{8}$$

where $\mathcal{P}_{k-1} = \frac{1}{\alpha_1\beta_1}P_{k-1}$ and P_{k-1} is a polynomial of degree $k-1$. As a result,

$$y - \mathbb{A}x_k = y - \mathbb{A}\mathbb{A}^*\mathcal{P}_{k-1}(\mathbb{A}\mathbb{A}^*)y = \mathcal{Q}_k(\mathbb{A}\mathbb{A}^*)y,$$

where $\mathcal{Q}_k(t) = 1 - t\mathcal{P}_{k-1}(t)$, a polynomial of degree k. Let (η_l, ϕ_l, ψ_l) be a singular system of $\mathbb{A}$, i.e., there exist orthonormal sequences $\{\phi_l\}$ in $\mathbb{X}$ and $\{\psi_l\}$ in $\mathbb{Y}$ such that

$$\mathbb{A}\phi_l = \eta_l\psi_l, \qquad \mathbb{A}^*\psi_l = \eta_l\phi_l, \tag{9}$$

for all $l \in \mathbb{N}$. If $y \in \text{span}\{\psi_1, ..., \psi_N\}$ for some $N \in \mathbb{N}$, i.e.,

$$y = \sum_{n=1}^{N} \gamma_l\psi_l,$$

then equations (8) and (9) result in

$$x_k = \sum_{n=1}^{N} \gamma_l\mathcal{P}_{k-1}(\eta_l^2)\eta_l\phi_l, \tag{10}$$

that means $x_k \in \text{span}\{\phi_1, ..., \phi_N\}$. Therefore, for a linear equation in finite dimensional Hilbert space, $\mathcal{LSAM}$ is guaranteed to converge in at most the dimension of the space in the absence of round-off error. Like Lemma 2.22 of [19], one can prove that the polynomial $\mathcal{Q}_k$ minimizes the functional

$$\mathcal{H}(\mathcal{Q}) := \| \mathcal{Q}(\mathbb{A}\mathbb{A}^*)y \|^2 \qquad \text{on} \qquad \{\mathcal{Q} \in \mathbb{P}_k : \mathcal{Q}(0) = 1\},$$

and

$$\mathcal{H}(\mathcal{Q}_k) = \| \mathbb{A}x_k - y \|^2,$$

where $\mathbb{P}_k$ denotes the space of polynomials of degree at most k. Also

$$\langle\langle \mathcal{Q}_i, \mathcal{Q}_j \rangle\rangle := \sum_{n=1}^{\infty} \eta_l^2\mathcal{Q}_i(\eta_l^2)\mathcal{Q}_j(\eta_l^2)|\langle y, \psi_l\rangle|^2 = 0, \qquad i \neq j.$$

$\langle\langle ., . \rangle\rangle$ defines an inner product on the space $\mathbb{P}$ if $y \notin \text{span}\{\psi_1, ..., \psi_N\}$ for any $N \in \mathbb{N}$. As a result, we have the following theorem.

Theorem 3 *Let $\mathbb{A}$ and $\mathbb{A}^*$ be injective and assume that $\mathcal{LSAM}$ does not stop after finitely many steps. Also let x_k be the sequence of the approximate solutions produced by $\mathcal{LSAM}$. Then*

$$\| \mathbb{A}x_k - y \| \longrightarrow 0 \qquad as \qquad k \to \infty,$$

for every $y \in \mathbb{Y}$.

For equations in infinite dimensional spaces, the convergence of $\mathcal{LSAM}$ requires a priori information. But Theorem 3 shows that the sequence of residuals converges to zero.

In what follows, we study the regularization effects of $\mathcal{LSAM}$. For this end, we consider (1) from the knowledge of a perturbed right hand side $y = y^{exact} + e$ where

$$\| e \| \leq \delta.$$

The Picard theorem [19,20] states that the solution of Eq. (1) is given by

$$x = \sum_{n=1}^{\infty} \frac{1}{\eta_l} \langle y, \psi_l \rangle \phi_l. \tag{11}$$

This formula indicates that the ill-posedness of (1) stems from the fact that the singular values η_l always decay to zero. On this basis, many regularization methods are based on filtering out the influence of $\frac{1}{\eta_l}$ in Eq. (11) as follows

$$x^{regular} = \sum_{n=1}^{\infty} f_l \frac{1}{\eta_l} \langle y, \psi_l \rangle \phi_l,$$

where f_l are the filter factors. We refer to [14,15] for further reading on filter factors and their properties. From Eq. (10), $\mathcal{LSAM}$ is a spectral filtering regularization method with filter factors

$$f_l = \mathcal{P}_{k-1}(\eta_l^2)\eta_l.$$

To obtain more insight into the filter factors of $\mathcal{LSAM}$, we need the following definition of Ritz values for a compact operator, introduced in [9, Theorem 8.1.15] and also in [28] for symmetric matrices.

Definition 3 Let $\mathbb{A} : \mathbb{X} \to \mathbb{X}$ be a compact and self-adjoin operator. The Ritz values $\vartheta_1, \vartheta_2, \ldots, \vartheta_k$ with respect to k dimensional subspace K of $\mathbb{X}$ are the eigenvalues of the mapping

$$\mathcal{A} := P\mathbb{A}|_K, \tag{12}$$

where P is the orthogonal projection operator from $\mathbb{X}$ onto K. Also the orthonormal eigenvectors of $\mathcal{A}$ corresponding to Ritz values are called Ritz vectors.

Now, let $\vartheta_1^{(k)}, \vartheta_2^{(k)}, \ldots, \vartheta_k^{(k)}$ be the Ritz values of $\mathbb{A}^*\mathbb{A}$ with respect to V_k and $y_1^{(k)}, y_2^{(k)}, \ldots, y_k^{(k)}$ be the associated Ritz vectors. We have the following Lemma, shown in [28] for a matrix.

Lemma 2 *It holds that*

$$\langle y_i^{(k)}, \mathbb{A}^* y \rangle \neq 0, \quad i = 1, 2, \ldots, k.$$

Proof: From Definition 3, it is obvious that

$$\mathbb{A}^*\mathbb{A}y_i^{(k)} - \vartheta_i^{(k)} y_i^{(k)} \perp V_k, \quad i = 1, 2, \ldots, k.$$

Hence for $j \leq k$,

$$\langle \mathbb{A}^*\mathbb{A}y_i^{(k)} - \vartheta_i^{(k)} y_i^{(k)}, (\mathbb{A}^*\mathbb{A})^{j-1}\mathbb{A}^* y \rangle = 0,$$

or equivalently

$$\langle y_i^{(k)}, (\mathbb{A}^*\mathbb{A})^j \mathbb{A}^* y \rangle = \vartheta_i^{(k)} \langle y_i^{(k)}, (\mathbb{A}^*\mathbb{A})^{j-1}\mathbb{A}^* y \rangle.$$

By repeating the above equality with replaced j by $j-1$ and continuing this process, we get

$$\langle y_i^{(k)}, (\mathbb{A}^*\mathbb{A})^j \mathbb{A}^* y \rangle = \vartheta_i^{(k)^j} \langle y_i^{(k)}, \mathbb{A}^* y \rangle.$$

As a result, if $\langle y_i^{(k)}, \mathbb{A}^* y\rangle = 0$, then $y_i^{(k)} \perp V_k$ which is contradicts that the set of Ritz vectors $y_i^{(k)}$ forms a basis for V_k. $\square$

By Eqs. (7) and (8),

$$x - x_k = \mathcal{Q}_k(\mathbb{A}^*\mathbb{A})x. \tag{13}$$

On the other hand, since x_k minimizes the residual $\|\mathbb{A}^* y - \mathbb{A}^*\mathbb{A}x\|$ on V_k, we have

$$\langle \mathbb{A}^*\mathbb{A}(x_k - x), v\rangle = 0, \quad v \in V_k.$$

Thus $\langle \mathbb{A}^*\mathbb{A}\mathcal{Q}_k(\mathbb{A}^*\mathbb{A})x, v\rangle = 0$ or $\langle \mathcal{Q}_k(\mathbb{A}^*\mathbb{A})\mathbb{A}^* y, v\rangle = 0$ for all $v \in V_k$. As a result, $P\mathcal{Q}_k(\mathbb{A}^*\mathbb{A})\mathbb{A}^* y = 0$, which is equivalent to

$$\mathcal{Q}_k(A_k)\mathbb{A}^* y = 0, \tag{14}$$

where $A_k = P\mathbb{A}^*\mathbb{A}$. Also, since $\{y_i^{(k)}\}$ is the orthonormal basis for V_k, we have

$$\mathbb{A}^* y = \sum_{i=1}^{k} \langle \mathbb{A}^* y, y_i^{(k)}\rangle y_i^{(k)}. \tag{15}$$

Substituting (15) in (14) and using Definition 3, we obtain

$$\sum_{i=1}^{k} \langle \mathbb{A}^* y, y_i^{(k)}\rangle \mathcal{Q}_k(\vartheta_i^{(k)}) y_i^{(k)} = 0$$

$$\Rightarrow \langle \mathbb{A}^* y, y_i^{(k)}\rangle \mathcal{Q}_k(\vartheta_i^{(k)}) = 0, \quad i = 1, \ldots, k.$$

Therefore, by Lemma 2

$$\mathcal{Q}_k(\vartheta_i^{(k)}) = 0, \quad i = 1, \ldots, k.$$

This means that the polynomial $\mathcal{Q}_k$ has the form

$$\mathcal{Q}_k(t) = \frac{(\vartheta_1^{(k)} - t)(\vartheta_2^{(k)} - t)\ldots(\vartheta_k^{(k)} - t)}{\vartheta_1^{(k)}\vartheta_2^{(k)}\ldots\vartheta_k^{(k)}}.$$

Replacing the above expression in (13) and using singular system, the following theorem is proved.

Theorem 4 *The filter factors associated with the kth iterate x_k of $\mathcal{LSAM}$ are as follows*

$$f_l = 1 - \prod_{i=1}^{k} \frac{\vartheta_i^{(k)} - \sigma_l^2}{\vartheta_i^{(k)}}.$$

From Theorem 4, the spectral filtering of $\mathcal{LSAM}$ is controlled by the Ritz values and similar to that of [15, Section 6.3.2], discussed for LSQR, can be seen whose convergence is related to the number of iteration k. In fact, k plays the role of regularization parameter that should be chosen properly by a suitable stopping rule. Before discussing about the stopping rule and perturbation analysis, we show that the Ritz values of $\mathbb{A}^*\mathbb{A}$ with respect to V_k are the same as the eigenvalues of $\mathbb{G}_k^T\mathbb{G}_k$. From Definition 3, the Ritz vectors of $\mathbb{A}^*\mathbb{A}$ are obviously in the form

$$y_i^{(k)} = \mathbb{V}_k \circ Z_i^{(k)}, \qquad i = 1, 2, \ldots k,$$

with $\mathbb{V}_k$ and $\circ$ as before. Hence, (12) results in

$$\begin{aligned} P\mathbb{A}^*\mathbb{A}\mathbb{V}_k \circ Z_i &= \vartheta_i^{(k)}\mathbb{V}_k \circ Z_i \\ &\Rightarrow P(\mathbb{V}_k \circ \mathbb{G}_k^T + \alpha_{k+1} v_{k+1} e_{k+1}^T) \circ (\mathbb{G}_k Z_i) = \vartheta_i^{(k)}\mathbb{V}_k \circ Z_i \\ &\Rightarrow P\mathbb{V}_k \circ \mathbb{G}_k^T\mathbb{G}_k Z_i = \vartheta_i^{(k)}\mathbb{V}_k \circ Z_i. \end{aligned}$$

The second equality results from using (3) and property (6) and the third one comes from properties of the projection operator P. Since the components of $\mathbb{V}_k$ are linearly independent, from the last equality, we have

$$\mathbb{G}_k^T\mathbb{G}_k Z_i = \vartheta_i^{(k)} Z_i, \qquad i = 1, 2, \ldots, k.$$

Concerning the convergence properties of the Ritz values, a similar discussion to that of LSQR, studied in [15, Section 6.4], can be done for $\mathcal{LSAM}$.

We conclude this section, with a theorem from [19] about perturbation analysis of the CG algorithm for Eq. (7) which is also true for $\mathcal{LSAM}$. We use the following stopping rule which is the discrepancy principle in the context of the iterative methods [7,19,20].

Stopping rule: Fix $\ell > 1$ and terminate the algorithm at the first time, $k = k(\delta)$, that

$$|\bar{\mu}_k| \leq \ell\delta,$$

where $\bar{\mu}_k$, norm of kth residual, is the same as the one in $\mathcal{LSAM}$.

If we assume that $y \notin \text{span}\{\psi_1, \psi_2, ..., \psi_N\}$ for any $N \in \mathbb{N}$, then by Theorem 3, $\|\mathbb{A}x_k - y\| \to 0$ as $k \to \infty$, i.e., the above stopping rule is well defined for $\mathcal{LSAM}$. To show optimality of $\mathcal{LSAM}$, as [19] we introduce the following normed spaces of $\mathbb{X}$

$$(\mathbb{A}^*\mathbb{A})^{\frac{\nu}{2}}(\mathbb{X}) := \{x \in \mathbb{X} \quad s.t. \quad \sum_{n=1}^{\infty} \eta_l^{-2\nu} |\langle x, \phi_l \rangle|^2 < \infty\},$$

with

$$\| x \|_\nu := \sqrt{\sum_{n=1}^{\infty} \eta_l^{-2\nu} |\langle x, \phi_l \rangle|^2},$$

where $\nu \geq 0$. The following theorem shows that $\mathcal{LSAM}$ with discrepancy principle is of optimal order under some smoothness conditions.

Theorem 5 *Let $y^{exact}, y \notin span\{\psi_1, \psi_2, ..., \psi_N\}$ for any $N \in \mathbb{N}$ and let $y \in (\mathbb{A}^*\mathbb{A})^{\frac{\nu}{2}}(\mathbb{X})$ for some $\nu > 0$ and $\| x \|_\nu \leq L \in \mathbb{R}$. If $\mathcal{LSAM}$ is stopped after $k(\delta)$ steps according to mentioned stopping rule with fixed parameter $\ell > 1$, then there exists $c > 0$ such that*

$$\|x_{k(\delta)}^{\delta} - x\| \leq cL^{\frac{1}{\nu+1}} \delta^{\frac{\nu}{\nu+1}}.$$

4 Test problem

In this section, we employ $\mathcal{LSAM}$ to solve the system of linear Fredholm integral equations of the first kind.

The system of Fredholm integral equations of the first kind have attracted much concern in applied sciences which is given by

$$\int_{\Gamma} \mathcal{K}(t,s)\mathcal{X}(s)ds = \mathcal{Y}(t), \qquad t \in \Gamma = [a,b], \tag{16}$$

with

$$\begin{aligned} \mathcal{X}(t) &= [x_1(t), x_2(t), ..., x_n(t)]^T, \\ \mathcal{Y}(t) &= [y_1(t), y_2(t), ..., y_n(t)]^T, \\ \mathcal{K}(t,s) &= [K_{ij}(t,s)], \qquad i,j = 1,2,...,n, \end{aligned}$$

where the kernel $\mathcal{K}$ and right hand side $\mathcal{Y}$ are given and $\mathcal{X}$ is the solution to be determined. For more details about this kind of problems, we refer to [30] and references therein.

We define the Hilbert space $\mathbb{H}$ and its inner product as follows

$$\mathbb{H} = \underbrace{L^2(\Gamma) \times L^2(\Gamma) \times ... \times L^2(\Gamma)}_{n \text{ times}}, \qquad \langle \Phi, \Psi \rangle_{\mathbb{H}} = \int_{\Gamma} \Phi(t)^H \Psi(t) dt, \qquad \Phi, \Psi \in \mathbb{H},$$

where $L^2(\Gamma)$ is the vector space of all functions $x : L^2(\Gamma) \to \mathbb{C}$ satisfying

$$\int_{\Gamma} |x(t)|^2 dt < \infty,$$

and Φ^H denotes the conjugate and transpose of Φ. Then the operator form of (16) is as follows

$$\mathbb{K}\mathcal{X} = \mathcal{Y}, \tag{17}$$

with

$$\mathbb{K} \ : \ \mathbb{H} \to \mathbb{H},$$
$$(\mathbb{K}\mathcal{X})(t) = \int_{\Gamma} \mathcal{K}(t,s)\mathcal{X}(s)ds.$$

Now, for employing $\mathcal{LSAM}$ to solve Equation (17), we need the adjoint of the operator $\mathbb{K}$. To fulfill this requirement, we can state the following theorem.

Theorem 6 *Let the individual kernels K_{ij} in $\mathcal{K} = [K_{ij}]_{1\leq i,j\leq n}$ be continuous or weakly singular, i.e., K_{ij} is continuous for all $t, s \in \Gamma$, $t \neq s$, and there exist positive constants M and $\alpha \in (0, 1]$ such that*

$$|K_{ij}(t,s)| \leq M|t-s|^{\alpha-1}, \qquad t,s \in \Gamma, \quad t \neq s.$$

Then in the dual system $\langle \mathbb{H}, \mathbb{H} \rangle$ the adjoint of the operator

$$(\mathbb{K}\Phi)(t) = \int_{\Gamma} \mathcal{K}(t,s)\Phi(s)ds, \tag{18}$$

is

$$(\mathbb{K}^*\Psi)(t) = \int_{\Gamma} \mathcal{K}(s,t)^H\Psi(s)ds.$$

Proof: The proof is trivial and follows from general theory of systems of integral equations in operator form in Hilbert space [1,20].

4.1 Discrete $\mathcal{LSAM}$

In $\mathcal{LSAM}$ for solving the system of linear integral equations of the first kind, computation of the parameters α_i, β_i and v_i, u_i involves some definite integrals. In this section, we look at the effect of replacing the integrals with numerical one. Two types of integrals appear in $\mathcal{LSAM}$: the integral operator $\mathbb{K}$ and the norm $\| \ . \ \|_{\mathbb{H}}$ related to the inner product $\langle ., . \rangle_{\mathbb{H}}$. To approximate these integrals, we use some convergent quadrature rule

$$\int_{\Gamma} \Phi(t)dt \approx \sum_{j=0}^{N} w_j \Phi(t_j), \qquad \Phi \in \mathbb{H}, \qquad t_j \in \Gamma, \ j = 0, 1, ..., N,$$

where $w_j > 0$, $j = 0, 1, ..., N$. As a result, $\mathbb{K}$ is replaced by $\mathbb{K}_N$ defined by

$$(\mathbb{K}_N\Phi)(t) = \sum_{j=0}^{N} w_j \mathcal{K}(t,t_j)\Phi(t_j), \qquad t \in \Gamma, \qquad \Phi \in \mathbb{H}.$$

In addition, the inner product $\langle ., . \rangle_{\mathbb{H}}$ and its corresponding norm $\| \ . \ \|_{\mathbb{H}}$ are replaced, respectively, by

$$\langle \Phi, \Psi \rangle_N = \sum_{j=0}^{N} w_j \Phi(t_j)^H \Psi(t_j), \qquad \Phi, \Psi \in \mathbb{H},$$

and

$$\| \ \Phi \ \|_N = \sqrt{\langle \Phi, \Phi \rangle_N}, \qquad \Phi \in \mathbb{H},$$

which is a discrete seminorm related to the discrete semidefinite inner product $\langle ., . \rangle_N$. We note that with assumption $N \geq k$ = number of iterations in $\mathcal{LSAM}$, $\|.\|_N$ is a seminorm on $\mathbb{H}$, and it is a norm on $\text{span}\{v_1, v_2, ..., v_k\}$.

The above discussion leads to a discrete $\mathcal{LSAM}$ that replaces operator equation (17) as

$$\mathbb{K}_N\mathcal{X} = \mathcal{Y}, \tag{19}$$

with

$$\mathbb{K}_N \ : \ \mathbb{H} \to \mathbb{H},$$
$$(\mathbb{K}_N \mathcal{X})(t) = \sum_{j=0}^{N} w_j \mathcal{K}(t, t_j)\mathcal{X}(t_j),$$

and Algorithm 1 is employed to solve approximate equation (19) with discrete norm $\|.\|_N$. It should be mentioned that the adjoint of the operator $\mathbb{K}_N$ with respect to inner product $\langle ., . \rangle_N$ is

$$(\mathbb{K}_N^* \Phi)(t) = \sum_{j=0}^{N} w_j \mathcal{K}(t_j, t)^H \Phi(t_j), \qquad \Phi \in \mathbb{H},$$

which is a discrete form of the adjoint of operator $\mathbb{K}$ with respect to inner product $\langle ., . \rangle_{\mathbb{H}}$:

$$(\mathbb{K}^* \Psi)(t) = \int_{\Gamma} \mathcal{K}(s, t)^H \Psi(s) ds, \qquad \Psi \in \mathbb{H}.$$

Now we describe the discrete $\mathcal{LSAM}$ for the case that the individual kernels are weakly singular, i.e., instead of K_{ik} in $\mathcal{K} = [K_{ik}]_{1 \le i,k \le n}$, we have

$$W(|t - s|)K_{ik}(t, s), \tag{20}$$

where $W : (0, \infty) \to \mathbb{R}$ is continuous and satisfies $|W(x)| \le Mx^{\alpha - 1}$ for all $x > 0$ and some positive constants M and α. Also, the remaining parts K_{ik}, $i, k = 1, \ldots, n$ are continuous on $\Gamma \times \Gamma$. For this end, we approximate the definite integrals in the operator $\mathbb{K}$ by using a quadrature rule

$$\int_{\Gamma} W(|t - s|)\Phi(s) ds \approx \sum_{j=0}^{N} w_j(t)\Phi(t_j), \qquad \Phi \in \mathbb{H}, \qquad t_j \in \Gamma, \ j = 0, 1, ..., N,$$

where quadrature weights $w_j(t)$, $j = 0, 1, \ldots, N$ depend continuously on t. Then the weakly singular integral operator $\mathbb{K}$ is replaced by

$$(\mathbb{K}_N \Phi)(t) = \sum_{j=0}^{N} w_j(t) \mathcal{K}(t, t_j)\Phi(t_j), \qquad t \in \Gamma. \tag{21}$$

Discrete semidefinite inner product and corresponding discrete seminorm for this case are different from continuous ones. We define them as follows

$$\langle \Phi, \Psi \rangle_N = \sum_{j=0}^{N} \Phi(t_j)^H \Psi(t_j), \qquad \| \Phi \|_N = \sqrt{\langle \Phi, \Phi \rangle_N}, \qquad \Phi, \Psi \in \mathbb{H},$$

and employ $\mathcal{LSAM}$ to approximate equation

$$\mathbb{K}_N \mathcal{X} = \mathcal{Y},$$

with the new discrete seminorm. We note that the adjoint of the operator $\mathbb{K}_N$ with respect to the new discrete semidefinite inner product is

$$(\mathbb{K}_N^* \Psi)(t) = \sum_{j=0}^{N} w_j(t) \mathcal{K}(t_j, t)^H \Psi(t_j), \qquad \Psi \in \mathbb{H},$$

which is the discrete form of the adjoint of operator $\mathbb{K}$ with respect to inner product $\langle ., . \rangle_{\mathbb{H}}$:

$$(\mathbb{K}^* \Psi)(t) = \int_{\Gamma} W(|t - s|) \mathcal{K}(s, t)^H \Psi(s) ds, \qquad \Psi \in \mathbb{H}.$$

A similar discussion about the case $W_{ik}(|t - s|)K_{ik}(t, s)$, $i, k = 1, \ldots, n$, instead of (20), can be done.

5 Numerical experiments

In this section, we give several examples to examine the proposed method numerically. Some of them have non-unique solution to assess the method in terms of computing minimum norm solution. In this regards, we compare $\mathcal{LSAM}$ with Tikhonov's regularization method, a method to find the minimum norm solution. Also, we show the regularization features of the discrete $\mathcal{LSAM}$ and compare it with the LSQR method when it is performed to obtain a regularized solution of the final system of equations obtained from the Nyström-like discretization. All the examples presented in this section were computed in double precision with a MATLAB code. The first two examples are constructed artificially. For these examples, first like the Nyström method we discretize the system of integral equations by a quadrature rule and compute the least norm solution of the obtained system of algebraic equations. And then we apply $\mathcal{LSAM}$ to solve the system of integral equations when the same quadrature rule is used to approximate the involved integrals and compare the results with the Nyström method.

Example 1 We consider the system (16) with

$$\mathcal{K}(t,s) = \begin{pmatrix} e^{ts} & (t^2+s^2)^{\frac{1}{2}} \\ \cos ts & e^{t+s} \end{pmatrix}, \tag{22}$$

$$\mathcal{Y}(t) = [\frac{(t^2+1)^{\frac{3}{2}}}{3} - \frac{t^3}{3} + \frac{e^{t+1}-1}{t+1} \quad e^t + \frac{e^1\cos t + e^1 t\sin t - 1}{t^2+1}]^T, \tag{23}$$

$\Gamma = [0,1]$ and exact solution $\mathcal{X}^*(t) = [e^t \quad t]^T$.

Example 2 We consider the system (16) with

$$\mathcal{K}(t,s) = \begin{pmatrix} 1-t+s & -t-s \\ t-s & -2t+s \end{pmatrix}, \tag{24}$$

$$\mathcal{Y}(t) = [\frac{1}{3} - t \quad \frac{-t}{2} + \frac{1}{6}]^T, \tag{25}$$

$\Gamma = [0,1]$. It is easy to see that $\mathbb{K}(\mathbb{H})$ is the three dimensional space

$$\text{span}\{[1 \quad -1]^T, [1-t \quad t]^T, [t \quad 2t]^T\},$$

and $\mathbb{A}$ maps its range bijectively onto itself. Therefore, by [17] and [21] the minimum norm solution is $\mathcal{X}^*(t) = [1-t \quad t]^T$.

For these examples, first we approximate the integrals in the systems using the Simpson's rule with the quadrature points $t_j = \frac{j}{n},\ j = 0,1,2,...,n$. The following equations are obtained

$$\sum_{j=0}^{n} w_j K_{11}(t,t_j)x_1(t_j) + \sum_{j=0}^{n} w_j K_{12}(t,t_j)x_2(t_j) = f_1(t), \tag{26}$$

$$\sum_{j=0}^{n} w_j K_{21}(t,t_j)x_1(t_j) + \sum_{j=0}^{n} w_j K_{22}(t,t_j)x_2(t_j) = f_2(t), \tag{27}$$

where

$$w_j = \begin{cases} \frac{1}{3n} & j = 0, n, \\ \frac{4}{3n} & j = 1,3,5,...,n-1, \\ \frac{2}{3n} & j = 2,4,...,n. \end{cases}$$

Now let t run through the nodal points $t_i,\ i = 0,1,...,n$. Then $\mathcal{X}(t_j) = [x_1(t_j), x_2(t_j)]^T,\ j = 0,1,...,n$ are approximated by solving the obtained linear system. Here, we have used MATLAB *pinv* function, returning the Moore-Penrose pseudoinverse, with the tolerance 10^{-6} to approximate the minimum norm solution of the final system. The absolute error in some points are shown in Table 1 for Example 1 with $n = 100$. The condition number of linear system for this example is $5.27\times10^{+19}$. We see that the errors are approximately large. This is due to the fact that the corresponding operator $\mathbb{K}$ does not have a bounded inverse and the condition numbers of its finite dimensional approximations grow with the quality of the

Table 1: Numerical results for the Example 1.

t	$\mid x_1(t) - x^*(t) \mid$	$\mid x_2(t) - x^*(t) \mid$
0	6.900×10^{-1}	4.72×10^{-3}
0.25	2.44×10^{-1}	5.23×10^{-2}
0.5	6.57×10^{-1}	2.00×10^{-1}
0.75	4.35×10^{-1}	1.54×10^{-1}
1	1.14×10^{0}	4.19×10^{-1}

Table 2: Numerical results for the Example 2.

t	$\mid x_1(t) - x^*(t) \mid$	$\mid x_2(t) - x^*(t) \mid$
0	7.00×10^{-1}	1.77×10^{-4}
0.25	1.50×10^{-1}	4.84×10^{-2}
0.5	2.00×10^{-1}	2.01×10^{-1}
0.25	5.20×10^{-2}	1.47×10^{-1}
1	1.44×10^{-3}	4.02×10^{-1}

approximation. For Example 2 with $n = 100$, the numerical results are reported in Table 2. The condition number of linear system for this example is 2.24×10^{19}. Let $\widetilde{\mathcal{X}}$ denote the obtained approximated solution and $\|.\|_N$ be, as in Section 4, the Sympson approximation of $\|.\|_{\mathbb{H}}$ with $N = 100$. The relative error norm, $\|\widetilde{\mathcal{X}} - \mathcal{X}^*\|_N / \|\mathcal{X}^*\|_N$, for both Examples 1 and 2 is equal to 2.89×10^{-1}. Now we apply $\mathcal{LSAM}$ for solving these examples with the same quadrature rules for approximating the involved definite integrals in each iteration.

Table 3 shows the numerical results for Example 1. In this table, $k^* = 96$ is the optimal index where the smallest relative error occurs. Moreover, Figure 1 plots the approximate solutions, obtained at the optimal iteration, of $\mathcal{LSAM}$, and the exact solutions. In tables and figures, x_i^*, $x_i^{\mathcal{L}}$ and "Iter" denote the ith component of the exact solution, the approximate solution obtained by $\mathcal{LSAM}$ and the number of iterations, respectively. We observe that the results have been improved considerably which is due to the regularizing property of $\mathcal{LSAM}$. Also the approximate solution

Table 3: Numerical results for the Example 1.

(a)

t	Iter	$\mid x_1^{\mathcal{L}}(t) - x_1^*(t) \mid$	$\mid x_2^{\mathcal{L}}(t) - x_2^*(t) \mid$
	8	5.04×10^{-3}	1.08×10^{-1}
0	16	2.24×10^{-2}	1.39×10^{-2}
	32	3.53×10^{-3}	9.94×10^{-3}
	8	7.34×10^{-3}	3.25×10^{-2}
0.25	16	8.61×10^{-3}	4.35×10^{-3}
	32	7.91×10^{-4}	6.93×10^{-3}
	8	1.84×10^{-3}	3.16×10^{-2}
0.5	16	2.19×10^{-3}	1.48×10^{-3}
	32	5.81×10^{-4}	7.23×10^{-3}
	8	6.70×10^{-3}	2.71×10^{-2}
0.75	16	9.27×10^{-3}	3.19×10^{-2}
	32	3.29×10^{-4}	8.99×10^{-3}
	8	2.90×10^{-3}	9.71×10^{-2}
1	16	2.16×10^{-2}	3.49×10^{-2}
	32	3.60×10^{-4}	2.31×10^{-2}

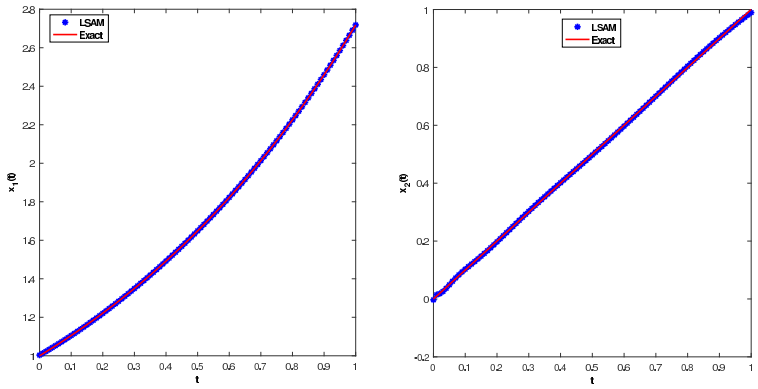
(b)

Iter	$\|\mathcal{X}^{\mathcal{L}} - \mathcal{X}^*\|_N / \|\mathcal{X}^*\|_N$
8	1.98×10^{-2}
16	5.81×10^{-3}
32	2.3×10^{-3}
$96(k^*)$	6.28×10^{-4}

$$\mathcal{X}^{\mathcal{L}}(t) = [1 + 8 \times 10^{-17} - (1 + 3 \times 10^{-17})t \;\; (1 + 3 \times 10^{-17})t - 2 \times 10^{-17}]^T,$$

is obtained after 3 iterations of $\mathcal{LSAM}$ when it is used to solve Example 2. We see that it is a very good approximation of the minimum norm solution $\mathcal{X}^*$. The associated results to Examples 1 and 2 show that we should distinguish between applying straightly of $\mathcal{LSAM}$ with numerically approximating involved integrals and applying LSQR to solve final system of algebraic equations, obtained from the Nyström method.

The next two examples are chosen from [21] to compare $\mathcal{LSAM}$ with the Tikhonov's method in terms



(a) $x_1^{\mathcal{L}}$ and x_1^* (b) $x_2^{\mathcal{L}}$ and x_2^*

Fig. 1: The exact and approximate solutions for example 1.

of computing the least norm solution when there is more than one solution. For the Tikhonov's method, we find the optimal regularization parameter experimentally. Also, we use these examples to show the regularization properties of $\mathcal{LSAM}$ and to compare with the regularized LSQR method in the discrete case. In these examples, $\Gamma = [0, 1]$.

Example 3 [21] We consider the system (16) with

$$\mathcal{K}(t, s) = (t - s)^2, \qquad \mathcal{Y}(t) = \frac{1}{4} - \frac{2t}{3} + \frac{t^2}{2},$$

with the minimum norm solution $\mathcal{X}^*(t) = t$.

Example 4 [21] We consider the system (16) with

$$\mathcal{K}(t, s) = \begin{cases} s^2(1 - t)^2(2st + s - 3t)/6, & s \leq t, \\ t^2(1 - s)^2(2st + t - 3s)/6, & s > t, \end{cases}$$

$$\mathcal{Y}(t) = \frac{(\frac{-163}{5040} + \frac{29u}{420} - \frac{u^2}{24} + \frac{u^3}{180} - \frac{u^4}{1680})}{16},$$

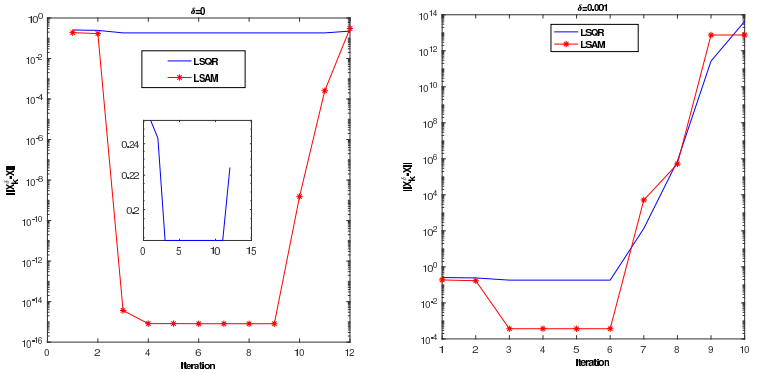
with minimum norm solution $\mathcal{X}^*(t) = (u - 1)^2$, where $u = (2t - 1)^2$.

For these examples, we use Simpson's rule with $n = 1000$ to discretize the Tikhonov's equation and approximate the involved definite integrals in $\mathcal{LSAM}$. The associated results to $\mathcal{LSAM}$ and the Tikhonov's regularization method, to compute the minimum norm solution, are presented in Table 4. In this table, $\mathcal{X}^{\mathcal{L}}$ and $\mathcal{X}^T$ denote the solutions obtained from $\mathcal{LSAM}$ and the Tikhonov's regularization method respectively.
Now we use the above examples to examine the regularization properties of $\mathcal{LSAM}$.

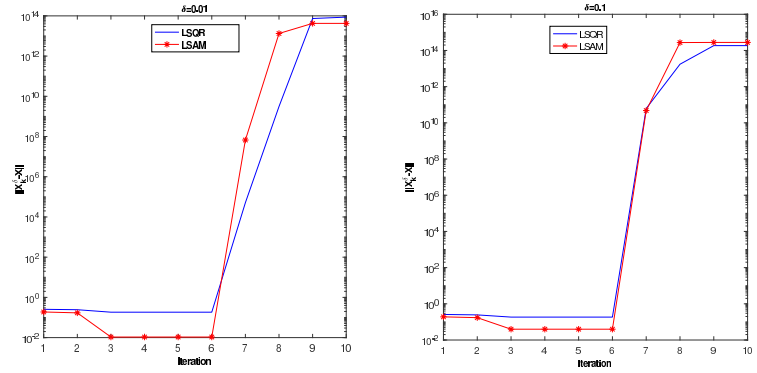
Table 4: $\mathcal{LSAM}$ and Tikhonov's regularization method for Examples 3, 4.

Example	$\| \mathcal{X}^{\mathcal{L}} - \mathcal{X}^* \|$	$\| \mathcal{X}^T - \mathcal{X}^* \|$
3	8.19×10^{-16}	1.07×10^{-7}
4	2.55×10^{-9}	1.69×10^{-6}

To this end, we compute the solution in quadrature points of numerical integration and investigate the influence of perturbation on this vector. In addition, we present the regularization manner of LSQR iteration method when it is applied to final system of linear equations arising from Nyström method based on the same quadrature rule as discrete $\mathcal{LSAM}$. We use Simpson's rule with quadrature points t_j, $j = 0, 1, \ldots, n$ and $n = 1000$ again. The condition number of linear system for Examples 3 and 4 are



(a) $\delta = 0$ (b) $\delta = 0.001$



(c) $\delta = 0.01$ (d) $\delta = 0.1$

Fig. 2: The errors in solutions for various δ.

equal to 4.10×10^{21} and ∞, respectively.

To report the numerical results, we use the following discrete norm

$$\|X\| = \sqrt{\frac{1}{n+1}\sum_{i=0}^{n} x_i^2}, \qquad X = [x_0 \ x_1 \ \dots \ x_n]^T \in \mathbb{R}^{n+1},$$

and perturb the discrete right hand side $Y = [\mathcal{Y}(t_0) \ \dots \ \mathcal{Y}(t_n)]^T$ by a normal distributed random error with $\mu = 0$ and standard deviation σ on Y with δ such that

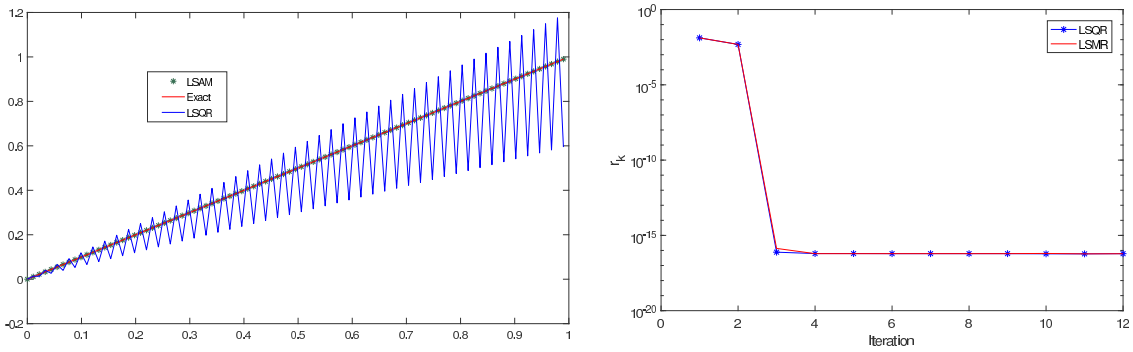
$$\|Y^\delta - Y\| \leq \delta,$$

where δ is a known noise level and Y^δ is perturbed right hand side.

Let X_k^δ and X be the kth approximate solution and exact solution, respectively, in quadrature points.

The numerical results for Example 3 are shown in Figures 2 and 3. In figure 2, for both LSQR and $\mathcal{LSAM}$, the error in solution $\|X_k^\delta - X\|$ is plotted versus the iteration number k for $\delta = 0.001$, $\delta = 0.01$, $\delta = 0.1$ and $\delta = 0.0$. We see that the smallest errors for LSQR are considerably large even for exact data. This is because of the fact that the LSQR method converges to the least Euclidian norm solution of the final linear system of equations which may be far from the original solution. While, $\mathcal{LSAM}$ is implemented on the original problem for obtaining the least L^2-norm solution. Figure 3 compares the $\mathcal{LSAM}$ and LSQR solutions with the exact solution for $\delta = 0$. We see that the $\mathcal{LSAM}$ gives a very good approximations of the exact solution for both examples. While the LSQR solution is a non-smooth perturbation of the exact one. Also, in part (b) of Figure 3 the residual norm (r_k) for $\delta = 0$ is plotted versus the iteration index k.

Table 5 lists the smallest solution errors for various δ for both of the Examples 3 and 4. In this table, we see that the smallest errors for $\mathcal{LSAM}$ are considerably smaller than LSQR. Moreover, for Example 3 the errors for $\mathcal{LSAM}$ increase by factor 10 when δ increases by factor 10. For Example 4, this



(a) Comparison of solutions for $\delta = 0$ (b) The residual norm for $\delta = 0$

Fig. 3: Numerical results for Example 3 in the case of exact data.

Table 5: The minimum of the errors in the solution for various δ for the Examples 3 and 4.

Example	Method	$\delta = 0$	$\delta = 0.001$	$\delta = 0.01$	$\delta = 0.1$
3	LSQR	1.84×10^{-1}	1.84×10^{-1}	1.84×10^{-1}	1.92×10^{-1}
	$\mathcal{LSAM}$	8.09×10^{-16}	5.82×10^{-4}	5.82×10^{-3}	5.82×10^{-2}
4	LSQR	2.02×10^{-1}	2.02×10^{-1}	2.03×10^{-1}	3.54×10^{-1}
	$\mathcal{LSAM}$	2.55×10^{-9}	1.03×10^{-2}	3.89×10^{-2}	3.79×10^{-1}

factor is almost equal to $10^{\frac{2}{3}}$ when δ increases from 0.001 to 0.01 and is almost 10 when δ increases from 0.01 to 0.1. These results confirm the Theorem 5. While the errors for the LSQR method remain almost unchanged for both of the examples.

Conclusions

We have proposed an iterative method, $\mathcal{LSAM}$, to compute the minimum norm solution of ill-posed problems. In addition, the regularization properties of $\mathcal{LSAM}$ under discrepancy principle in context of iterative methods, have been investigated. The proposed algorithm have been tested by applying it to solve the system of linear integral equations of the first kind and satisfactory results have been obtained from experiments. In addition, $\mathcal{LSAM}$ have been compared with Tikhonov's regularization method and numerical examples have shown that $\mathcal{LSAM}$ is more reliable. To examine the regularizing features of the presented algorithm, we have investigated the influence of perturbation on vector of solution in quadrature points of numerical integration rule used in discrete version of $\mathcal{LSAM}$. In this regards, the proposed method have been compared with the LSQR method when it is used to solve the final linear system of equations obtained from the Nyström method.

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