Numerical studies of a non-local parabolic partial differential equations by spectral collocation method with preconditioning

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Abstract

In this paper, the spectral collocation method with preconditioning is applied to solve a non-local parabolic partial differential equations. The cubic spline interpolation is implemented for approximating the non-local boundary condition. Two examples are given to illustrate the effectiveness of the method.

Key words: Spectral collocation method, preconditioning, non-local boundary conditions, cubic spline interpolation, fourth order Runge-Kutta.
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1. Introduction

An area of increasing scientific interest over the past decades is the study of the partial differential equations with nonlocal boundary conditions. One could generically classify these problems into two types: boundary value problems with nonlocal initial conditions, and boundary value problems with nonlocal boundary conditions [1, 2, 3, 5, 4, 6]. In the present paper, the second group of these nonlocal boundary value problems is considered.

Several papers have been presented in the literature to investigate the solution of partial differential equations with nonlocal boundary conditions [7, 8, 9, 10]. Dehghan in [11] applied some numerical schemes to approximate

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the solution of a parabolic equation with non-local boundary specifications. The Galerkin method has been applied to solve a similar equation by Bouziani [12].

This work is aimed at applying a very efficient method for solving the following non-local boundary value problem

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = f(x,t), \qquad a < x < b, \quad 0 < t \leq T \tag{1}$$

with the initial condition

$$u(x,0) = p(x), \qquad a \leq x \leq b, \tag{2}$$

and the boundary condition

$$u(0,t) = g(t), \qquad 0 < t \leq T, \tag{3}$$

and the non-local boundary condition

$$\int_a^b u(x,t)dx = m(t). \tag{4}$$

The functions $f(x,t)$, $p(x)$, $g(t)$ and $m(t)$ and the parameter α are known. In [13], a matrix formulation technique with arbitrary polynomial basis bases has been proposed for the numerical/analytical solutions of this kind of partial differential equations. Tatari and Dehghan applied the Gaussian radial basis functions (GA-RBF) to find the approximate solution for Eqs. (1)-(4) [14].

Spectral collocation methods for solving partial differential equations have become increasingly popular because of their ability to achieve high accuracy using relatively few grid points [15, 16, 17, 18, 19, 20, 21, 29]. In [22], the authors applied the spectral collocation method for computing an approximate solution to the generalized Hirota-Satsuma coupled KdV equation. Convergence of the spectral methods for Burgers' equations and singular differential equation is discussed in [23, 24]. In [25], Kim is discussed about the Chebyshev spectral collocation method with preconditioning for elliptic partial differential equations. Darvishi and Javidi in [26], applied the spectral collocation method with a preconditioning technique to solve Burger's equations.

In this paper, we apply the spectral collocation method with Darvishi's preconditioning for computing an approximate solution to (1)-(4). A technique is proposed to implement the non-local boundary condition in this method.

This paper is organized as follows. Section 2 is devoted to application of the spectral collocation method for solving equations (1)-(4). In section 3, some numerical results are presented to demonstrate the efficiency of the method. Some concluding remarks are presented in section 4.

2. Method discussion

In this section, we have been a brief description of the spectral collocation method or pseudospectral method, then the proposed method has been applied to Eq. (1)-(4).

The spectral collocation or pseudospectral methods are usually applied to approximate the function $f(x)$ with a truncated series expansion under some basis ψ_n

$$f(x) \approx f_N(x) = \sum_{j=0}^{N} a_n \psi_j(x). \tag{5}$$

The approximate solution will be found by solving for the spectral coefficients a_n. By assigning the collocation points to x, then by solving the system, the good results are obtained.

Consider the Nth-order Chebyshev polynomials

$$T_N(x) = \cos(N \cos^{-1}(x))$$

and suppose that $f(x)$ is a function on $[-1, 1]$. In [22, 26, 27], the authors used an interpolation of the function $f(x)$ by the polynomial $f_N(x)$ of degree at most N of the form

$$f_N(x) = \sum_{j=0}^{N} g_j(x) f(x_j), \tag{6}$$

in the Chebyshev-Gauss-Lobatto points

$$x_j = \cos \frac{j\pi}{N}, \quad j = 0, 1, \ldots, N, \tag{7}$$

where $g_j(x)$ for $j = 0, 1, \dots, N$ are polynomials of degree at most N which are obtained as follows:

$$g_j(x) = \frac{(-1)^{i+j}(1-x^2)T_N'(x)}{c_j N^2 (x - x_j)}, \quad j = 0, \dots, N, \tag{8}$$

such that

$$c_0 = c_N = 2, \qquad c_j = 1, \quad j = 0, 1, \dots, N. \tag{9}$$

The derivatives of approximate solution $f_N(x)$ are then estimated at the collocation points by differentiating (6) and evaluating the resulting expression [27, 28]. The derivatives are given as

$$f_N^{(n)}(x) = \sum_{j=0}^{N} g_j^{(n)}(x) f(x_j), \qquad n = 1, 2, \dots. \tag{10}$$

The first derivative of $f_N(x)$ at x_k becomes

$$f_N'(x_k) = \sum_{j=0}^{N} d_{kj}^{(1)} f(x_j), \qquad k = 0, 1, \dots, N, \tag{11}$$

or in matrix notation

$$F^{(1)} = D^{(1)} F,$$

where $D^{(1)}$ is the Chebyshev differential matrix and the entries of this matrix are given as

$$d_{kj}^{(1)} = \begin{cases} -\dfrac{c_k}{2c_j} \dfrac{(-1)^{(i+j)}}{\sin\left((k+j)\dfrac{\pi}{2N}\right) \sin\left((k-j)\dfrac{\pi}{2N}\right)}, & k \neq j \\ \dfrac{1}{2} \cos\left(\dfrac{k\pi}{N}\right)\left(1 + \cot^2\left(\dfrac{k\pi}{N}\right)\right), & k = j, \quad k \neq 0, N, \\ d_{00} = -d_{NN} = \dfrac{2N^2+1}{6}. \end{cases} \tag{12}$$

The second derivative matrix can also be given explicitly [15, 16], and here the relation $D^{(p)} = (D^{(1)})^p$ holds, which is not true for all collocation methods [16].

When k is near j, the values of $|d_{kj}|$ become larger, then we have a large ammount of the round-off error. To reduce the round-off error at kth node, Darvishi [28] defined a preconditioning technique as follows.

Let

$$h_k(x) = f_N(x) - f_N(x_k) \tag{13}$$

From (6), we have

$$h_k(x) = \sum_{j=0}^{N} g_j(x) h_k(x_j).$$

Thus

$$h_k^{(n)}(x) = \sum_{j=0}^{N} g_j^{(n)}(x) h_k(x_j).$$

Now, from Eq. (13) we see that $h_k^{(n)}(x) = f_N^{(n)}(x)$. Therefore, we obtain

$$\begin{aligned} f_N^{(n)}(x_k) = h_k^{(n)}(x_k) &= \sum_{j=0}^{N} g_j^{(n)}(x_k) h_k(x_j) \\ &= \sum_{j=0}^{N} d_{kj}^{(n)}(f(x_j) - f(x_k)) \end{aligned} \tag{14}$$

Now, we apply the proposed method to solve Eq. (1). The Chebyshev-Gauss-Lobatto points in the interval $[a, b]$ are

$$x_j = \frac{b-a}{2} \cos \frac{j\pi}{N} + \frac{b+a}{2}, \quad j = 0, 1, \ldots, N.$$

To solve Eq. (1) using pseudospectral method we discretize the equation in space and apply Eq. (14):

$$\begin{aligned} u_t(x_i, t) = \alpha \left(\frac{2}{b-a}\right)^2 \sum_{j=0}^{N-1} d_{ij}^{(2)} \left(u(x_j, t) - u(x_i, t)\right) \\ +\alpha \left(\frac{2}{b-a}\right)^2 d_{iN}^{(2)} \left(g(t) - u(x_i, t)\right) + f(x_i, t), \quad i = 0, 1, \ldots, N-1, \end{aligned} \tag{15}$$

with the following initial condition

$$u(x_i, 0) = p(x_i), \qquad i = 0, 1, \ldots, N. \tag{16}$$

Here, we mention that for our problem we have $x_N = a < x_{N-1} < \ldots < x_1 < x_0 = b$.

Let the solution of the problem is known at the grid points of the time level j. Note that the solution is known at time level $j = 0$. We use the fourth order Runge-Kutta method to solve the system of ordinary differential equations (15). After computing the solution of the problem at time level $j+1$, we modify the solution at the grid point (b, t_{j+1}). To do so, let $s_m(x)$ be the natural spline interpolation of degree d of u at grid points (x_i, t_{j+1}), $i = 1, 2, \ldots, N$. Now, we define the function p as

$$p(x) = s_d(x) + \beta \prod_{j=1}^{N} (x - x_j), \tag{17}$$

where $\beta \in \mathbb{R}$ has to be computed. Obviously, for every $\beta \in \mathbb{R}$, we have $p(x_i) = u(x_i, t_{j+1})$, $i = 1, \ldots, N$. Therefore, the function p can be considered as an approximation of the function u at the time level $t = t_{j+1}$. Now, considering the non-local condition (4) we compute the parameter β such a way that

$$\int_a^b p(x)dx = m(t).$$

This gives

$$\beta = \frac{m(x) - \displaystyle\int_a^b s_d(x)dx}{\displaystyle\int_a^b \prod_{j=1}^{N} (x - x_j)dx}. \tag{18}$$

After computing β, by using Eq. (17) we set $u(x_0, t_{j+1}) = p(x_0)$.

3. Numerical tests

In this section, the numerical method described in Sections 2 is used to solve two problems. The proposed numerical procedure uses a fourth-order Runge-Kutta method with $k = 0.0001$ in time and spectral collocation method with preconditioning with $N = 8$ in space. In Eq. (17), it is assumed that $d = 13$. The proposed method is implemented in the powerful MAPLE software.

Example 1. For the first example, we consider Eqs. (1)-(4) with

$$\begin{aligned} &f(x,t) = (x^2-2)e^t, \qquad 0<x<1, \quad 0<t\le T, \\ &p(x) = x^2, \qquad 0\le x\le 1 \\ &g(t) = 0, \qquad 0<t\le T, \\ &\alpha = 1, \quad a=0, \quad b=1, \quad m(t)=\tfrac{e^t}{3} \qquad 0<t\le T. \end{aligned} \tag{19}$$

The theoretical solution of this problem is

$$u(x,t) = x^2 e^t. \tag{20}$$

The absolute and relative errors at grid points of the computed solution are given for different values of time levels in Tables 1-2. As the numerical results in this tables show the proposed method is very effective.

Example 2. Tatari and Dehghan in [14] have considered equation (1)-(4) with

$$\begin{aligned} &f(x,t) = 0, \qquad 0<x<1, \\ &p(x) = \cos\left(\tfrac{\pi}{2}x\right), \qquad 0<x<1 \\ &g(t) = \exp\left(-\tfrac{\pi^2}{4}t\right), \qquad 0<t<1, \\ &\alpha = 1, \quad b=1, \quad m(t)=\tfrac{2}{\pi}\exp\left(-\tfrac{\pi^2}{4}t\right), \qquad 0<t<1. \end{aligned} \tag{21}$$

the exact solution of this problem is

$$u(x,t) = \exp\left(-\frac{\pi^2}{4}t\right)\cos\left(\frac{\pi}{2}x\right). \tag{22}$$

Similar to the previous example, the values of absolute and relative error for different values of x and t are given in Tables 3-4. The obtained results are seen to be very reliable and accurate.

For more investigation, the absolute errors for $0<t\le 1$ for examples 1 and 2 are plotted in Fig. 1. As we observe, there is very good agreement between the approximate solution obtained by the spectral collocation method and the exact solution.

By comparing the numerical results presented for this example we see that our method is more effective than the method presented [14].

Table 1: Absolute error for various values of x and t with $N = 8$ for Example 1

$x_i \backslash t_j$	0.1	0.2	0.4	0.6	0.8	1
x_1	1.19e-08	1.09e-09	6.07e-10	1.13e-08	2.47e-10	6.00e-11
x_2	7.05e-10	2.81e-11	8.30e-10	1.16e-10	2.17e-10	6.72e-10
x_3	5.94e-11	6.35e-11	3.31e-11	4.52e-11	8.65e-11	5.73e-11
x_4	4.25e-11	3.80e-11	3.98e-11	7.29e-11	6.47e-11	8.01e-11
x_5	1.60e-11	2.54e-11	1.82e-11	3.53e-11	4.46e-11	4.51e-11
x_6	1.27e-11	1.34e-11	2.10e-11	2.52e-11	2.58e-11	3.38e-11
x_7	1.79e-12	6.71e-12	3.28e-12	8.08e-12	1.30e-11	1.09e-11
x_8	8.14e-12	1.43e-12	1.37e-11	1.04e-11	1.38e-13	1.19e-11

Table 2: Relative error for various values of x and t with $N = 8$ for Example 1

$x_i \backslash t_j$	0.1	0.2	0.4	0.6	0.8	1
x_1	1.08e-08	8.92e-10	4.07e-10	6.17e-09	1.11e-10	2.21e-11
x_2	6.89e-10	2.49e-11	6.01e-10	6.90e-11	1.06e-10	2.67e-10
x_3	7.37e-11	7.14e-11	3.05e-11	3.41e-11	5.34e-11	2.89e-11
x_4	8.04e-11	6.50e-11	5.59e-11	8.37e-11	6.08e-11	6.17e-11
x_5	5.79e-11	8.33e-11	4.87e-11	7.74e-11	8.01e-11	6.64e-11
x_6	1.20e-10	1.15e-10	1.48e-10	1.45e-10	1.22e-10	1.30e-10
x_7	7.54e-11	2.56e-10	1.02e-10	2.07e-10	2.71e-10	1.87e-10
x_8	5.08e-09	8.07e-10	6.33e-09	3.93e-09	4.27e-11	3.01e-09

Table 3: Absolute error for various values of x and t with $N = 8$ for Example 2

$x_i \backslash t_j$	0.1	0.2	0.4	0.6	0.8	1
x_1	4.02e-10	2.62e-09	1.05e-09	2.44e-09	2.88e-09	6.50e-11
x_2	5.09e-11	1.72e-10	2.51e-10	4.63e-11	2.28e-11	3.51e-12
x_3	2.57e-10	1.18e-11	1.00e-11	3.07e-11	2.68e-11	4.04e-12
x_4	1.04e-10	7.93e-11	5.15e-11	1.80e-11	1.37e-11	1.22e-11
x_5	2.97e-11	2.09e-11	1.26e-11	9.18e-12	4.57e-12	4.41e-12
x_6	1.81e-10	1.46e-10	8.80e-11	5.51e-11	3.21e-11	1.92e-11
x_7	2.07e-11	1.45e-11	8.33e-12	6.84e-12	3.25e-12	2.57e-12
x_8	3.21e-10	2.53e-10	1.53e-10	9.76e-11	5.66e-11	3.50e-11

Table 4: Relative error for various values of x and t with $N = 8$ for Example 2

$x_i \backslash t_j$	0.1	0.2	0.4	0.6	0.8	1
x_2	8.61e-09	4.71e-09	1.13e-08	3.40e-09	2.75e-09	6.93e-10
x_3	2.86e-10	8.44e-11	1.18e-10	5.92e-10	8.45e-10	2.09e-10
x_4	2.84e-10	2.79e-10	2.96e-10	1.69e-10	2.11e-10	3.08e-10
x_5	5.37e-11	4.85e-11	4.78e-11	5.70e-11	4.65e-11	7.36e-11
x_6	2.62e-10	2.69e-10	2.67e-10	2.74e-10	2.61e-10	2.55e-10
x_7	2.72e-11	2.43e-11	2.29e-11	3.09e-11	2.40e-11	3.11e-11
x_8	4.11e-10	4.15e-10	4.11e-10	4.30e-10	4.08e-10	4.14e-10

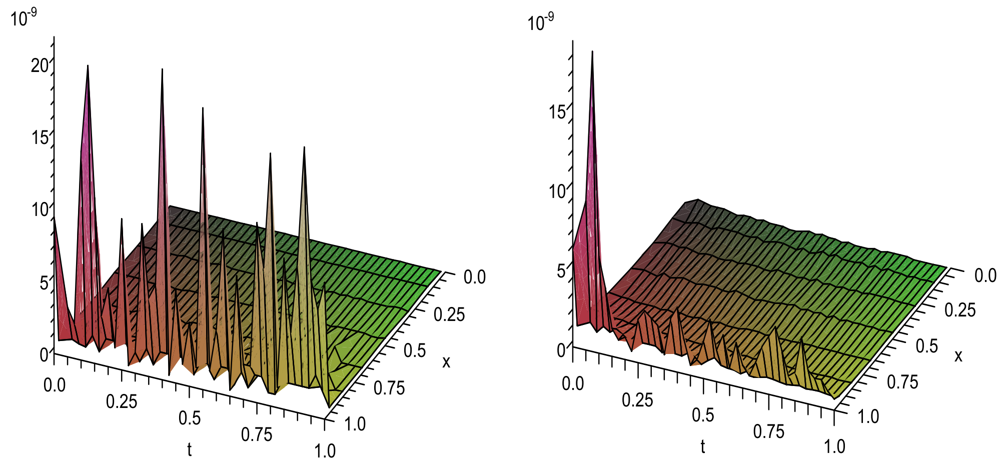

Figure 1: Absolute errors for $0 < t \leq 1$, Example 1 (left) and Example 2 (right) with $N = 8$.

4. Conclusion

In this paper, the spectral collocation method with preconditioning has been successfully used to obtain the approximate solutions to the non-local parabolic partial differential equations. To this end, a spectral collocation method in space and the fourth order Runge-Kutta scheme in time have been combined. The nonlocal boundary condition has been approximated by the cubic spline interpolation. Based on the numerical experiments, we conclude that our method is a practical and effective numerical technique for solving the non-local parabolic partial differential equations.

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