THE COUPLED SYLVESTER-TRANSPOSE MATRIX EQUATIONS OVER GENERALIZED CENTRO-SYMMETRIC MATRICES

Fatemeh Panjeh Ali Beik[§] **and Davod Khojasteh Salkuyeh**[*,†]

[§]*Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran*

e-mail: f.beik@vru.ac.ir, beik.fatemeh@gmail.com

[†]*Faculty of Mathematical Sciences, University of Guilan, P.O. Box 1914, Rasht, Iran*

e-mail: khojasteh@guialn.ac.ir, salkuyeh@gmail.com

Abstract. In this paper, we present an iterative algorithm for solving the following coupled Sylvester-transpose matrix equations

$$\sum_{j=1}^{q}\left(A_{ij}X_jB_{ij}+C_{ij}X_j^TD_{ij}\right)=F_i, \qquad i=1,2,\ldots,p,$$

over the generalized centro-symmetric matrix group $(X_1, X_2, \ldots, X_q)$. The solvability of the problem can be determined by the proposed algorithm, automatically. If the coupled Sylvester-transpose matrix equations are consistent over the generalized centro-symmetric matrices, then a generalized centro-symmetric solution group can be obtained within finite iterative steps for any initial generalized centro-symmetric matrix group in the exact arithmetic. Furthermore, it is shown that the least-norm generalized centro-symmetric solution group of the coupled Sylvester-transpose matrix equations can be computed by choosing an appropriate initial iterative matrix group. Moreover, the optimal approximate generalized centro-symmetric solution group to a given arbitrary matrix group $(V_1, V_2, \ldots, V_q)$ can be derived by finding the least-norm generalized centro-symmetric solution group of a new coupled Sylvester-transpose matrix equations. Finally, some numerical results are given to illustrate the validity and practicability of the theoretical results established in this work.

Keywords: Coupled Sylvester-transpose matrix equations, Iterative algorithm, Generalized centro-symmetric matrix.
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1. Introduction

Throughout this paper, we use $\mathrm{tr}(A), A^T, \overline{A}, \mathrm{Range}(A), \mathrm{Null}(A)$ to denote the trace, the transpose, the conjugate, the column space and the null space of the matrix A, respectively. Moreover, $\mathbb{R}^{m\times n}$ represents the set of all $m\times n$ real matrices and the set of all symmetric orthogonal matrices in $\mathbb{R}^{n\times n}$ is denoted by $\mathbb{SOR}^{n\times n}$. For two matrices Y and Z in $\mathbb{R}^{n\times s}$, the inner product $< Y, Z >_F$ is defined such that $< Y, Z >_F = \mathrm{tr}(Y^TZ)$.

[*]Corresponding author.

Definition 1.1. (Chen [5]) *For arbitrary given matrix $R \in \mathbb{SOR}^{n\times n}$, i.e., $R = R^T = R^{-1}$, we say that the matrix $A \in \mathbb{R}^{n\times n}$ is a generalized centro-symmetric matrix with respect to R, if $A = RAR$. The set of order n generalized centro-symmetric matrices with respect to R is denoted by $\mathbb{CSR}_R^{n\times n}$.*

Linear matrix equations play an important role in many areas, such as control theory, systems theory, stability theory and some other fields of pure and applied mathematics. In the literature, the problem of finding a solution to several linear matrix equations has been investigated widely, for more details see [3, 4, 10, 11, 13, 14, 17, 21, 24, 28] and the references therein. Before stating the main problems of this paper, we briefly review some of the works which have been recently presented in the field of linear matrix equations.

In [1], Beik and Salkuyeh have proposed two global Krylov subspace methods for solving the following general coupled linear matrix equations

$$\sum_{j=1}^{p} A_{ij}X_jB_{ij} = C_i, \qquad i = 1, 2, \ldots, p, \tag{1.1}$$

where $A_{ij} \in \mathbb{R}^{m\times m}$, $B_{ij} \in \mathbb{R}^{n\times n}$, $C_i \in \mathbb{R}^{m\times n}$, $i, j = 1, 2, \ldots, p$, are given matrices and $X_i \in \mathbb{R}^{m\times n}$, $i = 1, 2, \ldots, p$, are the unknown matrices. Zhang in [26] has extended the CGNE [20] and Bi-CGSTAB [20] algorithms to solve (1.1).

Song et al. [22] have considered the following coupled Sylvester-transpose matrix equations

$$\sum_{\eta=1}^{p} \left(A_{i\eta}X_\eta B_{i\eta} + C_{i\eta}X_\eta^T D_{i\eta}\right) = F_i, \qquad i = 1, 2, \ldots, N, \tag{1.2}$$

where $A_{i\eta} \in \mathbb{R}^{m_i\times l_\eta}$, $B_{i\eta} \in \mathbb{R}^{n_\eta\times p_i}$, $C_{i\eta} \in \mathbb{R}^{m_i\times n_\eta}$, $D_{i\eta} \in \mathbb{R}^{l_\eta\times p_i}$, $F_i \in \mathbb{R}^{m_i\times p_i}$, $i = 1, \ldots, N$, $\eta = 1, \ldots, p$, are given matrices, and $X_\eta \in \mathbb{R}^{l_\eta\times n_\eta}, \eta \in I[1, p]$ are the matrices to be determined. In the case that (1.2) has an unique solution, the authors have proposed an iterative algorithm by using the hierarchical identification principle.

Recently, the idea of the conjugate gradient (CG) method has been developed for constructing iterative algorithms to compute the solution of different kinds of linear matrix equations over reflexive and anti-reflexive, generalized bisymmetric, generalized centro-symmetric, mirror-symmetric, skew-symmetric and (P, Q)-reflexive matrices, for more details see [7, 8, 9, 12, 13, 17, 18, 19, 23, 25]. For instance, Dehghan and Hajarian [7] have presented an iterative method to solve the general coupled matrix equations (1.1) over the generalized bisymmetric matrix group $(X_1, X_2, \ldots, X_p)$, where $A_{ij} \in \mathbb{R}^{r_i\times n_j}$, $B_{ij} \in \mathbb{R}^{n_j\times s_i}$, and $C_i \in \mathbb{R}^{r_i\times s_i}$. In addition, more recently, Wu et al. [25] focused on the following coupled Sylvester-conjugate matrix equations

$$\sum_{\eta=1}^{p} \left(A_{i\eta}X_\eta B_{i\eta} + C_{i\eta}\overline{X_\eta}D_{i\eta}\right) = F_i, \qquad i = 1, 2, \ldots, p,$$

and proposed an iterative algorithm by extending the idea of CG method.

To the best of our knowledge, however, few results have been proposed in the literature for the following coupled Sylvester-transpose matrix equations

$$\sum_{j=1}^{q} \left(A_{ij}X_jB_{ij} + C_{ij}X_j^TD_{ij}\right) = F_i, \qquad i = 1, 2, \ldots, p. \tag{1.3}$$

where $A_{ij}, C_{ij} \in \mathbb{R}^{r_i\times n_j}$, $B_{ij}, D_{ij} \in \mathbb{R}^{n_j\times s_i}$, $F_i \in \mathbb{R}^{r_i\times s_i}$ are given and $X_j \in \mathbb{R}^{n_j\times n_j}$ for $j = 1, 2, \ldots, q$ are the unknown matrices. More precisely, the problem of finding the generalized centro-symmetric solution group to the above coupled linear matrix equations has not been considered so far. Matrix equations, that involve both unknown matrix and its transpose, arise in reducing a block anti-triangular matrix to a block anti-diagonal one, a reduction which is used in palindromic generalized eigenvalue problems [15]. It also appears in solving time-invariant Hamiltonian systems [16]. In [12], Hajarian and Dehghan have presented an algorithm to compute a generalized centro-symmetric solution to the special case of (1.3), namely, the linear matrix equation $AYB + CY^TD = E$. They have also presented an application of the mentioned linear matrix equation to a physical problem. In this paper, we are interested in constructing an algorithm for finding the solution of (1.3) over generalized centro-symmetric matrices.

In the following, we present a new product denoted by $\circledast$ which is efficient for establishing the theoretical results in Section 3.

Definition 1.2. *Let $A = [A_1, A_2, \ldots, A_k]$ and $B = [B_1, B_2, \ldots, B_k]$, where $A_i, B_i \in \mathbb{R}^{r_i\times s_i}$ for $i = 1, 2, \ldots, k$. Then the $k \times k$ matrix $A^T \circledast B$ is defined by*

$$A^T \circledast B = \mathrm{diag}\left(< A_1, B_1 >_F, \ldots, < A_k, B_k >_F\right).$$

Using the $\circledast$ product, we may define the following inner product and its corresponding matrix norm which are utilized for obtaining the main results of this work.

Definition 1.3. *Suppose that the matrix groups $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_k)$ and $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_k)$ are given where $\Phi_i, \Psi_i \in \mathbb{R}^{r_i\times s_i}$ for $i = 1, 2, \ldots, k$. We define the inner product $< ., . >$ as follows:*

$$< \Phi, \Psi >= \mathrm{tr}(\Phi^T \circledast \Psi). \tag{1.4}$$

Remark 1.4. *For $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_k)$, where $\Gamma_i \in \mathbb{R}^{r_i\times s_i}$, $i = 1, 2, \ldots, k$, the norm of Γ is defined by $\|\Gamma\|^2 = \mathrm{tr}(\Gamma^T \circledast \Gamma)$.*

1.1. **Problem reformulation.** In this work, we consider the following problems.

Problem I. For the given matrices $A_{ij}, C_{ij} \in \mathbb{R}^{r_i\times n_j}$, $B_{ij}, D_{ij} \in \mathbb{R}^{n_j\times s_i}$, $F_i \in \mathbb{R}^{r_i\times s_i}$ and $R_j \in \mathbb{SOR}^{n_j\times n_j}$, find the generalized centro-symmetric matrix group $X = (X_1, X_2, \ldots, X_q)$ with $X_j \in \mathbb{CSR}_{R_j}^{n_j\times n_j}$, $j = 1, 2, \ldots, q$, such that

$$\sum_{j=1}^{q} \left(A_{ij}X_jB_{ij} + C_{ij}X_j^TD_{ij}\right) = F_i, \qquad i = 1, 2, \ldots, p.$$

Problem II. Suppose that Problem I is consistent and S_r denotes the set of its solution groups. For a given arbitrary matrix group $V = (V_1, V_2, \ldots, V_q)$, find a generalized centro-symmetric matrix group $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_q)$ such that

$$\left\| V - \widetilde{X} \right\|^2 = \min_{X \in S_r} \|V - X\|^2 .$$

In fact, the main purpose of the second problem is to obtain the optimal approximate generalized centro-symmetric matrix group to a group of given matrices $V_j \in \mathbb{R}^{n_j \times n_j}$ for $j = 1, 2, \ldots, q$.

The rest of this paper is organized as follows. In Section 2, we first recall some necessary definitions and concepts. Then, we prove some preliminary results which are used for presenting the main results of this work. In Section 3, by developing the idea of the CG method, an algorithm is proposed to derive a generalized centro-symmetric solution group for the coupled Sylvester-transpose matrix equations. Theoretical results are established to show that if Problem I is consistent, then the proposed algorithm can compute a generalized centro-symmetric solution group to (1.3) within finite number of iteration steps in absence of roundoff errors. In addition, an approach is proposed for solving Problem II. Section 4 is devoted to presenting some numerical experiments to demonstrate the validity of the results presented in this work. Finally, the paper is ended with a brief conclusion in Section 5.

2. PRELIMINARIES

In this section, we review some principles and establish some results which are utilized in the next section.

The Kronecker product of the matrices $A = [a_{ij}]_{m \times s}$ and $B = [b_{ij}]_{n \times q}$ is defined as the $mn \times sq$ matrix $A \otimes B = [a_{ij} B]$. The "vec" operator transforms a matrix A of size $m \times s$ to a vector $a = vec(A)$ of size $ms \times 1$ by stacking the columns of A. In this paper, we frequently use the following relation (see [2])

$$\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X).$$

Lemma 2.1. *Let $X \in \mathbb{R}^{m \times n}$ be an arbitrary matrix. Then*

$$\text{vec}(X^T) = P(m, n)\text{vec}(X),$$

where $P(m, n)$ is uniquely determined by the integers m and n.

Proof. See [6]. □

Some properties of the matrix $P(m, n)$ are given as follows (see [6, 17, 27]):

1. For two arbitrary integer numbers m and n, $P(m, n)$ has the following explicit form

$$P(m,n) = \begin{pmatrix} E_{11}^T & E_{12}^T & \cdots & E_{1n}^T \\ E_{21}^T & E_{22}^T & \cdots & E_{2n}^T \\ \vdots & \vdots & \ddots & \vdots \\ E_{m1}^T & E_{m2}^T & \cdots & E_{mn}^T \end{pmatrix}_{mn\times mn},$$

where E_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ is an $m \times n$ matrix with the entry at position (i, j) being one and the others being zero.

2. For two arbitrary integer numbers m and n, $P(m, n)$ is an unitary matrix, i.e., $P(m, n)P^T(m, n) = P^T(m, n)P(m, n) = I_{mn}$.
3. For two arbitrary integer m and n, $P(m, n) = P^T(n, m)$.

2.1. **Useful linear operators and their relationships.** Throughout this paper, we utilize the following three linear operators for simplicity. The first linear operator is denoted by $\mathcal{M}$ and defined as follows:

$$\begin{aligned}&\mathcal{M} : \mathbb{R}^{n_1\times n_1} \times \cdots \times \mathbb{R}^{n_q\times n_q} \to \mathbb{R}^{r_1\times s_1} \times \cdots \times \mathbb{R}^{r_p\times s_p}\\ &X = (X_1, \ldots, X_q) \mapsto \mathcal{M}(X) = (M_1(X), \ldots, M_p(X)),\end{aligned}$$

where

$$M_i(X) = \sum_{j=1}^{q} \left(A_{ij}X_jB_{ij} + C_{ij}X_j^TD_{ij}\right), \qquad i = 1, 2, \ldots, p.$$

Therefore, using the linear operator $\mathcal{M}(X)$, the matrix equations (1.3) can be reformulated in the following form:

$$\mathcal{M}(X) = F,$$

where $F = (F_1, F_2, \ldots, F_p)$ and $F_i \in \mathbb{R}^{r_i\times s_i}$, $i = 1, 2, \ldots, p$.
The second linear operator is represented by $\mathcal{A}$ and defined as

$$\begin{aligned}&\mathcal{A} : \mathbb{R}^{r_1\times s_1} \times \cdots \times \mathbb{R}^{r_p\times s_p} \to \mathbb{R}^{n_1\times n_1} \times \cdots \times \mathbb{R}^{n_q\times n_q}\\ &Y = (Y_1, \ldots, Y_p) \mapsto \mathcal{A}(Y) = (A_1(Y), \ldots, A_q(Y)),\end{aligned}$$

where

$$A_j(Y) = \sum_{i=1}^{p} \left(A_{ij}^TY_iB_{ij}^T + D_{ij}Y_i^TC_{ij}\right), \qquad j = 1, 2, \ldots, q.$$

Moreover, we define the linear operator $\mathcal{D}$ as follows:

$$\begin{aligned}&\mathcal{D} : \mathbb{R}^{r_1\times s_1} \times \cdots \times \mathbb{R}^{r_p\times s_p} \to \mathbb{R}^{n_1\times n_1} \times \cdots \times \mathbb{R}^{n_q\times n_q}\\ &Y = (Y_1, \ldots, Y_p) \mapsto \mathcal{D}(Y) = (D_1(Y), \ldots, D_q(Y)),\end{aligned}$$

where

$$D_j(Y) = \frac{1}{2}\sum_{i=1}^{p} \left(A_{ij}^TY_iB_{ij}^T + D_{ij}Y_i^TC_{ij} + R_jA_{ij}^TY_iB_{ij}^TR_j + R_jD_{ij}Y_i^TC_{ij}R_j\right),$$

and the matrices $R_j \in \mathbb{SOR}^{n_j\times n_j}$, for $j = 1, 2, \ldots, q$, are given.

It is not difficult to establish the following proposition.

Proposition 2.2. *Let $X = (X_1, X_2, \ldots, X_q)$ and $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_q)$ be two given matrix groups, where $X_j, \widetilde{X}_j \in \mathbb{R}^{n_j \times n_j}$ for $j = 1, 2, \ldots, q$. Moreover, let the matrices $R_j \in \mathbb{SOR}^{n_j \times n_j}$, $j = 1, 2, \ldots, q$, be given. Assume that the matrix groups $Z = (Z_1, Z_2, \ldots, Z_q)$ and $\widetilde{Z} = (\widetilde{Z}_1, \widetilde{Z}_2, \ldots, \widetilde{Z}_q)$ are defined as $Z_j = R_j X_j R_j$ and $\widetilde{Z}_j = R_j \widetilde{X}_j R_j$ for $j = 1, 2, \ldots, q$. Then,*

$$< X, \widetilde{X} >=< Z, \widetilde{Z} > .$$

The following remarks are deduced from Proposition 2.2 immediately.

Remark 2.3. *In addition to the assumptions of Proposition 2.2, let $X_j \in \mathbb{CSR}_{R_j}^{n_j \times n_j}$ for $j = 1, 2, \ldots, q$. Then,*

$$< X, \widetilde{X} >=< X, \widetilde{Z} >,$$

where $\widetilde{Z}$ is defined as before.

Remark 2.4. *In addition to the assumptions of Proposition 2.2, let $X_j \in \mathbb{CSR}_{R_j}^{n_j \times n_j}$ and $\widetilde{X}_j = -R_j \widetilde{X}_j R_j$ for $j = 1, 2, \ldots, q$. Then,*

$$< X, \widetilde{X} >= 0.$$

Remark 2.5. *Let $X = (X_1, X_2, \ldots, X_q)$ and $Y = (Y_1, Y_2, \ldots, Y_p)$ be two matrix groups such that $X_j \in \mathbb{CSR}^{n_j \times n_j}$ and $Y_i \in \mathbb{R}^{r_i \times s_i}$ for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$. Moreover, assume that the matrices $R_j \in \mathbb{SOR}^{n_j \times n_j}$, $j = 1, 2, \ldots, q$, are given. Then,*

$$< \mathcal{A}(Y), X >=< \mathcal{D}(Y), X > .$$

Proposition 2.6. *Suppose that the matrix groups $X = (X_1, X_2, \ldots, X_q)$, $Y = (Y_1, Y_2, \ldots, Y_p)$ and $\mathcal{M}(X), \mathcal{A}(Y)$ are defined as before. Then,*

$$< \mathcal{M}(X), Y >=< X, \mathcal{A}(Y) > .$$

Proof. Straightforward computations show that

$$\begin{aligned} < \mathcal{M}(X), Y >&= \operatorname{tr}\left((\mathcal{M}(X))^T \circledast Y \right) \\ &= \sum_{i=1}^{p} \operatorname{tr}\left(\left(\sum_{j=1}^{q} B_{ij}^T X_j^T A_{ij}^T + D_{ij}^T X_j C_{ij}^T \right) Y_i \right) \\ &= \sum_{i=1}^{p} \sum_{j=1}^{q} \left(\operatorname{tr}\left(B_{ij}^T X_j^T A_{ij}^T Y_i \right) + \operatorname{tr}\left(D_{ij}^T X_j C_{ij}^T Y_i \right) \right) . \end{aligned}$$

Invoking the fact that for two arbitrary matrices A and B, $\operatorname{tr}(AB) = \operatorname{tr}(BA) = \operatorname{tr}(A^T B^T)$, we can rewrite the above relation as follows:

$$\begin{aligned} < \mathcal{M}(X), Y >&= \sum_{i=1}^{p} \sum_{j=1}^{q} \left(\operatorname{tr}\left(X_j^T A_{ij}^T Y_i B_{ij}^T \right) + \operatorname{tr}\left(X_j^T D_{ij} Y_i^T C_{ij} \right) \right) \\ &= \sum_{j=1}^{q} \operatorname{tr}\, (\; X_j^T (\sum_{i=1}^{p} A_{ij}^T Y_i B_{ij}^T + D_{ij} Y_i^T C_{ij}) \;) \\ &= \sum_{j=1}^{q} \operatorname{tr}(X_j^T A_j(Y)) = \operatorname{tr}(X^T \circledast \mathcal{A}(Y)) =< X, \mathcal{A}(Y) > . \end{aligned}$$

$\square$

3. Main results

In this section, we discuses on the solutions of Problems I and II.

3.1. **Proposed algorithm for Problem I.** In the following, we propose an algorithm for solving the coupled Sylvester-transpose matrix equations (1.3) over generalized centro-symmetric matrices by extending the idea of the conjugate gradient (CG) method [20]. One can compare the proposed algorithm step by step with the CG method to see that it is indeed an extension of the CG method.

Algorithm 1. *Algorithm for solving Problem I.*

1. Input matrices $A_{ij}, C_{ij} \in \mathbb{R}^{r_i \times n_j}$, $B_{ij}, D_{ij} \in \mathbb{R}^{n_j \times s_i}$, $F_i \in \mathbb{R}^{r_i \times s_i}$ and $R_j \in \mathbb{SOR}^{n_j \times n_j}$ for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$.
2. Choose arbitrary initial matrix group $X^{(1)} = (X_1^{(1)}, \ldots, X_q^{(1)})$ such that $X_j^{(1)} \in \mathbb{CSR}_{R_j}^{n_j \times n_j}$ for $j = 1, 2, \ldots, q$.
3. Calculate $R^{(1)} = F - \mathcal{M}(X^{(1)})$; Set $P^{(1)} = \mathcal{D}(R^{(1)})$ and $k = 1$.
4. If $R^{(k)} = 0$ or $R^{(k)} \neq 0$ and $P^{(k)} = 0$ then Stop; Otherwise go to 5.
5. Compute

$$X^{(k+1)} = X^{(k)} + \frac{\left\|R^{(k)}\right\|^2}{\left\|P^{(k)}\right\|^2} P^{(k)};$$

$$R^{(k+1)} = R^{(k)} - \frac{\left\|R^{(k)}\right\|^2}{\left\|P^{(k)}\right\|^2} \mathcal{M}(P^{(k)});$$

$$P^{(k+1)} = \mathcal{D}(R^{(k+1)}) + \frac{\left\|R^{(k+1)}\right\|^2}{\left\|R^{(k)}\right\|^2} P^{(k)}.$$

6. Go to Step 4.

For the matrix groups $P^{(k)} = (P_1^{(k)}, P_2^{(k)}, \ldots, P_q^{(k)})$ and $X^{(k)} = (X_1^{(k)}, X_2^{(k)}, \ldots, X_q^{(k)})$ produced by Algorithm 1, it is not difficult to see that $P_j^{(k)}, X_j^{(k)} \in \mathbb{CSR}_{R_j}^{n_j \times n_j}$, $j = 1, 2, \ldots, q$.

In the following, we analyze some properties of Algorithm 1. Then, it is shown that the algorithm stops after finite number of iterations in the exact arithmetic.

Lemma 3.1. *Assume that the sequences* $\{R^{(k)}\}$ *and* $\{P^{(k)}\}$, $k = 1, 2, \ldots, s$ ($R^{(k)} \neq 0$, $k = 1, 2, \ldots, s$) *are produced by Algorithm 1. Then,*

$$< R^{(i)}, R^{(j)} >= 0, \qquad < P^{(i)}, P^{(j)} >= 0, \qquad i, j = 1, 2, \ldots, s,\ i \neq j. \tag{3.1}$$

Proof. Without loss of generality, we only show that the assertion (3.1) is valid for $1 \leq i < j \leq k$. To do so, we use induction.

Step 1. For $k=2$, using Propositions 2.2 and 2.6, we get:

$$\begin{aligned} < R^{(1)}, R^{(2)} >= \left\|R^{(1)}\right\|^2 - \frac{\left\|R^{(1)}\right\|^2}{\left\|P^{(1)}\right\|^2} < \mathcal{M}(P^{(1)}), R^{(1)} > \\ = \left\|R^{(1)}\right\|^2 - \frac{\left\|R^{(1)}\right\|^2}{\left\|P^{(1)}\right\|^2} < P^{(1)}, \mathcal{A}(R^{(1)}) > \\ = \left\|R^{(1)}\right\|^2 - \frac{\left\|R^{(1)}\right\|^2}{\left\|P^{(1)}\right\|^2} < P^{(1)}, \mathcal{D}(R^{(1)}) > \\ = \left\|R^{(1)}\right\|^2 - \frac{\left\|R^{(1)}\right\|^2}{\left\|P^{(1)}\right\|^2} < P^{(1)}, P^{(1)} >= 0. \end{aligned}$$

Furthermore, it is easy to see that

$$\begin{aligned} < P^{(1)}, P^{(2)} >=< P^{(1)}, \mathcal{D}(R^{(2)}) + \frac{\left\|R^{(2)}\right\|^2}{\left\|R^{(1)}\right\|^2} P^{(1)} > \\ =< P^{(1)}, \mathcal{D}(R^{(2)}) > + \frac{\left\|R^{(2)}\right\|^2}{\left\|R^{(1)}\right\|^2} \left\|P^{(1)}\right\|^2 \\ =< \mathcal{M}(P^{(1)}), R^{(2)} > + \frac{\left\|R^{(2)}\right\|^2}{\left\|R^{(1)}\right\|^2} \left\|P^{(1)}\right\|^2 \\ = \frac{\left\|P^{(1)}\right\|^2}{\left\|R^{(1)}\right\|^2} < R^{(1)} - R^{(2)}, R^{(2)} > + \frac{\left\|R^{(2)}\right\|^2}{\left\|R^{(1)}\right\|^2} \left\|P^{(1)}\right\|^2 = 0. \end{aligned}$$

Step 2. Suppose that the conclusion (3.1) holds when $k=\ell$. We show that the conclusion (3.1) is valid for $k=\ell+1$.
It is not difficult to conclude that

$$\begin{aligned} < R^{(\ell)}, R^{(\ell+1)} > &= < R^{(\ell)}, R^{(\ell+1)} > \\ &= \left\|R^{(\ell)}\right\|^2 - \frac{\left\|R^{(\ell)}\right\|^2}{\left\|P^{(\ell)}\right\|^2} < R^{(\ell)}, \mathcal{M}(P^{(\ell)}) > \\ &= \left\|R^{(\ell)}\right\|^2 - \frac{\left\|R^{(\ell)}\right\|^2}{\left\|P^{(\ell)}\right\|^2} < \mathcal{A}(R^{(\ell)}), P^{(\ell)} > \\ &= \left\|R^{(\ell)}\right\|^2 - \frac{\left\|R^{(\ell)}\right\|^2}{\left\|P^{(\ell)}\right\|^2} < \mathcal{D}(R^{(\ell)}), P^{(\ell)} > \\ &= \left\|R^{(\ell)}\right\|^2 - \frac{\left\|R^{(\ell)}\right\|^2}{\left\|P^{(\ell)}\right\|^2} < P^{(\ell)} - \frac{\left\|R^{(\ell)}\right\|^2}{\left\|R^{(\ell-1)}\right\|^2} P^{(\ell-1)}, P^{(\ell)} > \\ &= 0. \end{aligned}$$

On the other hand, we deduce:

$$\begin{aligned}
< P^{(\ell)}, P^{(\ell+1)} > &= \; < P^{(\ell)}, P^{(\ell+1)} > \\
&= < P^{(\ell)}, \mathcal{D}(R^{(\ell+1)}) > + \frac{\left\|R^{(\ell+1)}\right\|^2}{\|R^{(\ell)}\|^2} \left\|P^{(\ell)}\right\|^2 \\
&= < P^{(\ell)}, \mathcal{A}(R^{(\ell+1)}) > + \frac{\left\|R^{(\ell+1)}\right\|^2}{\|R^{(\ell)}\|^2} \left\|P^{(\ell)}\right\|^2 \\
&= < \mathcal{M}(P^{(\ell)}), R^{(\ell+1)} > + \frac{\left\|R^{(\ell+1)}\right\|^2}{\|R^{(\ell)}\|^2} \left\|P^{(\ell)}\right\|^2 \\
&= \frac{\left\|P^{(\ell)}\right\|^2}{\|R^{(\ell)}\|^2} < R^{(\ell)} - R^{(\ell+1)}, R^{(\ell+1)} > + \frac{\left\|R^{(\ell+1)}\right\|^2}{\|R^{(\ell)}\|^2} \left\|P^{(\ell)}\right\|^2 \\
&= 0.
\end{aligned}$$

For $j = 1, 2, \ldots, \ell - 1$, we have:

$$\begin{aligned}
< R^{(j)}, R^{(\ell+1)} > &= \; < R^{(j)}, R^{(\ell)} - \frac{\left\|R^{(\ell)}\right\|^2}{\|P^{(\ell)}\|^2} \mathcal{M}(P^{(\ell)}) > \\
&= -\frac{\left\|R^{(\ell)}\right\|^2}{\|P^{(\ell)}\|^2} < R^{(j)}, \mathcal{M}(P^{(\ell)}) > \\
&= -\frac{\left\|R^{(\ell)}\right\|^2}{\|P^{(\ell)}\|^2} < \mathcal{A}(R^{(j)}), P^{(\ell)} > \\
&= -\frac{\left\|R^{(\ell)}\right\|^2}{\|P^{(\ell)}\|^2} < \mathcal{D}(R^{(j)}), P^{(\ell)} > \\
&= -\frac{\left\|R^{(\ell)}\right\|^2}{\|P^{(\ell)}\|^2} < P^{(j)} - \frac{\left\|R^{(j)}\right\|^2}{\|R^{(j-1)}\|^2} P^{(j-1)}, P^{(\ell)} > \\
&= 0.
\end{aligned}$$

and

$$\begin{aligned}
< P^{(j)}, P^{(\ell+1)} > &= \; < P^{(j)}, \mathcal{D}(R^{(\ell+1)}) + \frac{\left\|R^{(\ell+1)}\right\|^2}{\|R^{(\ell)}\|^2} P^{(\ell)} > \\
&= < P^{(j)}, \mathcal{D}(R^{(\ell+1)}) > \\
&= < P^{(j)}, \mathcal{A}(R^{(\ell+1)}) > \\
&= < \mathcal{M}(P^{(j)}), R^{(\ell+1)} > \\
&= \frac{\left\|P^{(j)}\right\|^2}{\|R^{(j)}\|^2} < R^{(j+1)} - R^{(j)}, R^{(\ell+1)} > \\
&= 0.
\end{aligned}$$

From Steps 1 and 2, the conclusion (3.1) holds by the principle of the induction. □

Lemma 3.2. *Suppose that Problem I is consistent. Moreover, assume that the matrix group $X^* = (X_1^*, X_2^*, \ldots, X_q^*)$ is an arbitrary solution of Problem I. Then for any initial generalized centro-symmetric matrix group $X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \ldots, X_q^{(1)})$, the sequences $X^{(i)}, R^{(i)}$ and $P^{(i)}$(produced by Algorithm 1) satisfy the following equality*

$$< P^{(i)}, X^* - X^{(i)} >= \|R^{(i)}\|^2, \qquad i = 1, 2, 3, \ldots \quad . \tag{3.2}$$

Proof. We prove the assertion (3.2) by induction. For $i = 1$, we derive:

$$\begin{aligned}
< P^{(1)}, X^* - X^{(1)} > &= < \mathcal{D}(R^{(1)}), X^* - X^{(1)} > \\
&= < \mathcal{A}(R^{(1)}), X^* - X^{(1)} > \\
&= < R^{(1)}, \mathcal{M}(X^* - X^{(1)}) > \\
&= < R^{(1)}, \mathcal{M}(X^*) - \mathcal{M}(X^{(1)}) > \\
&= < R^{(1)}, F - \mathcal{M}(X^{(1)}) > \\
&= < R^{(1)}, R^{(1)} > \\
&= \left\|R^{(1)}\right\|^2 .
\end{aligned}$$

Let the conclusion (3.2) be true for $i = \ell$. Then, for $i = \ell + 1$ we obtain:

$$\begin{aligned}
&< P^{(\ell+1)}, X^* - X^{(\ell+1)} > \\
&\quad =< \mathcal{D}(R^{(\ell+1)}), X^* - X^{(\ell+1)} > + \frac{\left\|R^{(\ell+1)}\right\|^2}{\|R^{(\ell)}\|^2} < P^{(\ell)}, X^* - X^{(\ell+1)} > \\
&\quad =< \mathcal{A}(R^{(\ell+1)}), X^* - X^{(\ell+1)} > + \frac{\left\|R^{(\ell+1)}\right\|^2}{\|R^{(\ell)}\|^2} < P^{(\ell)}, X^* - X^{(\ell+1)} > \\
&\quad =< R^{(\ell+1)}, \mathcal{M}(X^* - X^{(\ell+1)}) > + \frac{\left\|R^{(\ell+1)}\right\|^2}{\|R^{(\ell)}\|^2} < P^{(\ell)}, X^* - X^{(\ell+1)} > \\
&\quad =< R^{(\ell+1)}, R^{(\ell+1)} > + \frac{\left\|R^{(\ell+1)}\right\|^2}{\|R^{(\ell)}\|^2} \left(< P^{(\ell)}, X^* - X^{(\ell)} > - \frac{\left\|R^{(\ell)}\right\|^2}{\|P^{(\ell)}\|^2} < P^{(\ell)}, P^{(\ell)} > \right) \\
&\quad = \left\|R^{(\ell+1)}\right\|^2 + \frac{\left\|R^{(\ell+1)}\right\|^2}{\|R^{(\ell)}\|^2} \left(\left\|R^{(\ell)}\right\|^2 - \left\|R^{(\ell)}\right\|^2 \right) = \left\|R^{(\ell+1)}\right\|^2 .
\end{aligned}$$

By the principle of induction, the result (3.2) holds for $i = 1, 2, 3, \ldots$. □

Remark 3.3. *From Lemma 3.2, it is deduced that if there exists a positive integer number k such that $P^{(k)} = 0$ and $R^{(k)} \neq 0$, then the matrix equation* (1.3) *is not consistent over the generalized centro-symmetric matrices.*

Theorem 3.4. *Let Problem I be consistent. Then, for any initial generalized centro-symmetric matrix group $X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \ldots, X_q^{(1)})$ and in the absence of roundoff errors, a solution of Problem I can be computed by Algorithm 1 within at most $m + 1$ iteration steps, where $m = \sum_{i=1}^{p} r_i s_i$.*

Proof. Let $R^{(j)} \neq 0$, $j = 1, 2, \ldots, m$. From Remark 3.3, we conclude that $P^{(j)} \neq 0$ for $j = 1, 2, \ldots, m$. Therefore, $X^{(m+1)}$ and $R^{(m+1)}$ can be calculated by Algorithm 1. We denote $\mathcal{S}$ as a matrix subspace consists of the matrices of the form $E = (E_1, E_2, \ldots, E_p)$ where $E_i \in \mathbb{R}^{r_i \times s_i}$ for $i = 1, 2, \ldots, p$. As established by Lemma 3.1,

$$< R^{(i)}, R^{(j)} >= 0, \qquad i, j = 1, 2, \ldots, m+1, \qquad (i \neq j),$$

which shows that the set of matrices $R^{(i)}$, $i = 1, 2, \ldots, m$, is an orthogonal basis for $\mathcal{S}$. It is not difficult to see that $R^{(m+1)} = 0$, therefore $X^{(m+1)}$ is a solution of Problem I. □

Evidently, the coupled matrix equations (1.3) have a generalized centro-symmetric solution group $X = (X_1, X_2, \ldots, X_q)$, where $X_j \in \mathbb{CSR}^{n_j \times n_j}_{R_j}$ for $j = 1, 2, \ldots, q$, if and only if the system of matrix equations

$$\left\{ \begin{array}{l} \sum\limits_{j=1}^{q} \left(A_{ij} X_j B_{ij} + C_{ij} X_j^T D_{ij} \right) = F_i, \\ \sum\limits_{j=1}^{q} \left(A_{ij} R_j X_j R_j B_{ij} + C_{ij} R_j X_j^T R_j D_{ij} \right) = F_i \ , \end{array} \right. \tag{3.3}$$

is consistent, where $i = 1, 2, \ldots, p$, and the matrices $R_j \in \mathbb{SOR}^{n_j \times n_j}$ for $j = 1, 2, \ldots, q$ are given. Let us define the matrix Ψ by:

$$\Psi =$$

$$\begin{pmatrix} B_{11}^T \otimes A_{11} + (D_{11}^T \otimes C_{11}) P(n_1, n_1) & B_{12}^T \otimes A_{12} + (D_{12}^T \otimes C_{12}) P(n_2, n_2) & \cdots & B_{1q}^T \otimes A_{1q} + (D_{1p}^T \otimes C_{1q}) P(n_q, n_q) \\ \vdots & \vdots & & \vdots \\ B_{p1}^T \otimes A_{p1} + (D_{p1}^T \otimes C_{p1}) P(n_1, n_1) & B_{p2}^T \otimes A_{p2} + (D_{p2}^T \otimes C_{p2}) P(n_2, n_2) & \cdots & B_{pq}^T \otimes A_{pq} + (D_{pq}^T \otimes C_{pq}) P(n_q, n_q) \\ B_{11}^T R_1 \otimes A_{11} R_1 + (D_{11}^T R_1 \otimes C_{11} R_1) P(n_1, n_1) & B_{12}^T R_2 \otimes A_{12} R_2 + (D_{12}^T R_2 \otimes C_{12} R_2) P(n_2, n_2) & \cdots & B_{1q}^T R_q \otimes A_{1q} R_q + (D_{1q}^T R_q \otimes C_{1q} R_q) P(n_q, n_q) \\ \vdots & \vdots & & \vdots \\ B_{p1}^T R_1 \otimes A_{p1} R_1 + (D_{p1}^T R_1 \otimes C_{p1} R_1) P(n_1, n_1) & B_{p2}^T R_2 \otimes A_{p2} R_2 + (D_{p2}^T R_2 \otimes C_{p2} R_2) P(n_2, n_2) & \cdots & B_{pq}^T R_q \otimes A_{pq} R_q + (D_{pq}^T R_q \otimes C_{pq} R_q) P(n_q, n_q) \end{pmatrix}.$$

It is not difficult to see that Eq. (3.3) is equivalent to the following linear system of equations

$$\Psi \widehat{X} = \widehat{F},$$

where

$$\widehat{X} = \left(\mathrm{vec}(X_1)^T, \mathrm{vec}(X_2)^T, \ldots, \mathrm{vec}(X_q)^T \right)^T,$$

and

$$\widehat{F} = \left(\mathrm{vec}(F_1)^T, \mathrm{vec}(F_2)^T, \ldots, \mathrm{vec}(F_p)^T, \mathrm{vec}(F_1)^T, \mathrm{vec}(F_2)^T, \ldots, \mathrm{vec}(F_p)^T \right)^T.$$

Let $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_p)$ be an arbitrary matrix group, where $\Lambda_i \in \mathbb{R}^{r_i \times s_i}$ for $i = 1, 2, \ldots, p$. Corresponding to the matrix Λ, we define $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_q)$ such that $\Phi = 2\mathcal{D}(\Lambda)$. Therefore, it can be shown that

$$\widetilde{\Phi} = \Psi^T \widehat{\Lambda}, \tag{3.4}$$

where

$$\widetilde{\Phi} = (\text{vec}(\varphi_1)^T, \text{vec}(\varphi_2)^T, \ldots, \text{vec}(\varphi_q)^T)^T,$$

and

$$\widehat{\Lambda} = (\text{vec}(\Lambda_1)^T, \text{vec}(\Lambda_2)^T, \ldots, \text{vec}(\Lambda_p)^T, \text{vec}(\Lambda_1)^T, \text{vec}(\Lambda_2)^T, \ldots, \text{vec}(\Lambda_p)^T)^T.$$

From Eq. (3.4), we deduce that $\widetilde{\Phi} \in \text{Range}(\Psi^T)$. Consider the construction of the sequence $P^{(i)}$, $i = 1, 2, 3, ...$, in Algorithm 1 and let the initial matrix group be chosen such that $X^{(1)} = 2\mathcal{D}(\Lambda)$, as a special case $X^{(1)} = 2\mathcal{D}(0) = 0$. It is not difficult to see that the sequences $X^{(j)}$ produced by Algorithm 1 belongs to $\text{Range}(\Psi^T)$.

Assume that the linear system $Ax = b$ is consistent. It is well-known that the solutions of the linear system $Ax = b$ are computed by $x = A^\dagger b + \text{Null}(A)$, where $A^\dagger$ denotes the Moore-Penrose pseudoinverse of the matrix A [2]. Therefore, $A^\dagger b$ gives the minimum norm solution. Invoking the fact that $\text{Range}(A^\dagger) = \text{Range}(A^T)$, it is concluded that $x^* \in \text{Range}(A^T)$ has the smallest 2-norm solution of the linear system $Ax = b$. As a result, if the initial iterative matrix group $X^{(1)}$ is chosen such that

$$X^{(1)} = 2\mathcal{D}(\Lambda), \tag{3.5}$$

then the generalized centro-symmetric solution group constructed by Algorithm 1 is the least-norm generalized centro-symmetric solution group. We summarize this fact in the following theorem.

Theorem 3.5. *Let the Problem I be consistent. If $X^{(1)} = 2\mathcal{D}(\Lambda)$ where Λ is an arbitrary matrix group. Then the solution X^*, generated by Algorithm 1, is the least-norm generalized centro-symmetric solution group of the coupled matrix equation* (1.3).

3.2. **On the solution of Problem II.** This subsection is devoted to proposing an approach for solving Problem II.

Consider an arbitrary matrix group $V = (V_1, V_2, \ldots, V_q)$ where $V_j \in \mathbb{R}^{n_j \times n_j}$ for $j = 1, 2, \ldots, q$. Associated to the matrix group V, we define $W = (W_1, W_2, \ldots, W_q)$ and $\widehat{W} = (\widehat{W_1}, \widehat{W_2}, \ldots, \widehat{W_q})$ such that

$$W_j = \frac{1}{2}(V_j + R_j V_j R_j) \qquad \text{and} \qquad \widehat{W_j} = \frac{1}{2}(V_j - R_j V_j R_j),$$

where the matrices $R_j \in \mathbb{SOR}^{n_j \times n_j}$ are given and $j = 1, 2, \ldots, q$.

Assume that $X = (X_1, X_2, \ldots, X_q)$ is an arbitrary matrix group such that $X_j \in \mathbb{CSR}_{R_j}^{n_j \times n_j}$, $j = 1, 2, \ldots, q$. Since $W_j \in \mathbb{CSR}_{R_j}^{n_j \times n_j}$, $j = 1, 2, \ldots, q$, Remark 2.4 implies that

$$< \widehat{W}, X - W >= 0.$$

Hence, we derive:

$$\|V - X\|^2 = \|W - X\|^2 + \left\|\widehat{W}\right\|^2. \tag{3.6}$$

Therefore, Eq. (3.6) shows that

$$\min_{X \in S_r} \|V - X\|^2 = \min_{X \in S_r} \|W - X\|^2.$$

When the Problem I is consistent, then the set of its solution groups S_r is nonempty. For an arbitrary matrix group $V = (V_1, V_2, \ldots, V_q)$, straightforward computations show that the coupled Sylvester-transpose matrix equations

$$\sum_{j=1}^{q} \left(A_{ij} X_j B_{ij} + C_{ij} X_j^T D_{ij}\right) = F_i, \qquad i = 1, 2, \ldots, p,$$

is equivalent to the following coupled Sylvester-transpose matrix equations

$$\sum_{j=1}^{q} \left(A_{ij} Z_j B_{ij} + C_{ij} Z_j^T D_{ij}\right) = \widehat{F}_i, \qquad i = 1, 2, \ldots, p, \tag{3.7}$$

where

$$\widehat{F}_i = F_i - \sum_{j=1}^{q} \left(A_{ij} W_j B_{ij} + C_{ij} W_j^T D_{ij}\right), \qquad i = 1, 2, \ldots, p,$$

$Z_j = X_j - W_j$ and $W_j = \frac{1}{2}(V_j + R_j V_j R_j)$ for $j = 1, 2, \ldots, q$. In order to find the solution of Problem II, employing Algorithm 1 with the special initial matrix group of the form (3.5), we first find the least-norm generalized centro-symmetric solution group $Z^* = (Z_1^*, Z_2^*, \ldots, Z_q^*)$ of (3.7). Thus, the solution of Problem II can be represented as $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_q)$ where

$$\widetilde{X}_j = Z_j^* + W_j, \qquad j = 1, 2, \ldots, q.$$

4. Numerical experiments

In this section we present two numerical examples to show the effectiveness of the proposed algorithm. All the numerical experiments presented in this section were computed in double precision with some Matlab codes on a Pentium 4 PC, with a 3.06 GHz CPU and 1.00GB of RAM.

Example 4.1. We consider the coupled Sylvester-transpose matrix equations

$$\begin{cases} A_{11} X_1 B_{11} + C_{12} X_2^T D_{12} = F_1, \\ C_{21} X_1^T D_{21} + A_{22} X_2 B_{22} = F_2, \end{cases} \tag{4.1}$$

where

$$A_{11} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -3 & 1 \\ 3 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \quad B_{11} = \begin{pmatrix} 2 & 1 & 3 \\ -3 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix},$$

$$C_{12} = \begin{pmatrix} -1 & 2 & 2 & 4 \\ 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ -1 & 2 & 1 & 4 \\ 1 & -1 & 0 & 1 \end{pmatrix}, \quad D_{12} = \begin{pmatrix} 1 & 6 & 3 \\ 2 & 1 & 1 \\ -1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix},$$

$$C_{21} = \begin{pmatrix} 2 & 4 & -1 \\ 1 & -1 & 2 \\ 1 & 8 & 2 \\ 2 & 3 & 4 \end{pmatrix}, \quad D_{21} = \begin{pmatrix} 4 & -2 & 2 & 1 \\ 1 & 2 & -1 & 2 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} 2 & 1 & -4 & 3 \\ -1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 5 \\ 2 & 3 & 1 & 2 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 4 & -5 & 1 & 4 \\ 2 & 1 & 4 & -1 \\ -1 & 1 & -3 & 1 \\ -3 & 2 & 1 & 1 \end{pmatrix},$$

and

$$F_1 = \begin{pmatrix} 126 & -172 & 22 \\ 54 & 39 & 107 \\ 3 & -127 & -40 \\ 170 & -209 & 61 \\ 18 & 25 & -31 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 29 & -37 & 89 & -64 \\ -95 & 201 & 3 & -18 \\ -189 & 163 & -101 & -104 \\ 109 & 24 & 63 & 12 \end{pmatrix}.$$

The matrix equations (4.1) are consistent over the generalized centro-symmetric matrices and have the exact solution $X^* = (X_1^*, X_2^*)$ with

$$X_1^* = \begin{pmatrix} 5 & 2 & 7 \\ 6 & -3 & 0 \\ -1 & -4 & 1 \end{pmatrix} \in \mathbb{CSR}_{R_1}^{3\times 3}, \quad X_2^* = \begin{pmatrix} 8 & -7 & -1 & -8 \\ -2 & 10 & 2 & 2 \\ -2 & 0 & 12 & 2 \\ -10 & -2 & 2 & 10 \end{pmatrix} \in \mathbb{CSR}_{R_2}^{4\times 4},$$

where

$$R_1 = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & 2 \end{pmatrix} \in \mathbb{SOR}^{3\times 3}, \quad R_2 = \frac{1}{5}\begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 4 & -1 & -2 \\ 2 & -1 & 4 & -2 \\ 4 & -2 & -2 & 1 \end{pmatrix} \in \mathbb{SOR}^{4\times 4}.$$

We first solve Problem I corresponding to system (4.1) by Algorithm 1. The initial guess was taken to be the zero matrix group $X^{(1)} = (X_1^{(1)}, X_2^{(1)}) = (0, 0)$. Here, it

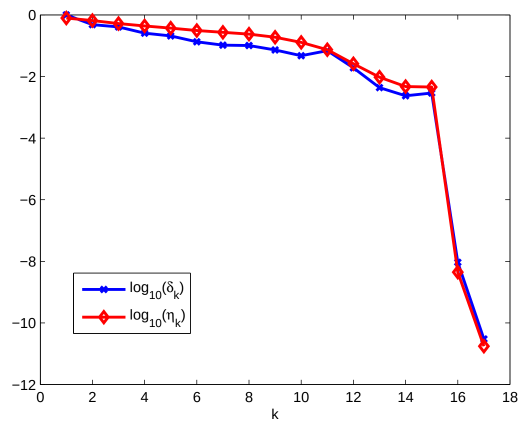

FIGURE 1. $\log_{10}\delta_k$ and $\log_{10}\eta_k$ versus k for Example 1 (Problem I).

is noted that the matrices $X_1^{(1)}$ and $X_2^{(1)}$ are generalized centro-symmetric. The used stopping criterion was

$$\delta_k = \frac{\|R^{(k)}\|}{\|R^{(1)}\|} < 10^{-10}.$$

In this case, the method converges in 17 iterations and provides the approximate solution $X^{(17)} = (X_1^{(17)}, X_2^{(17)})$ where

$$X_1^{(17)} = \begin{pmatrix} 4.99999999991379 & 1.99999999996927 & 7.00000000015604 \\ 6.00000000003797 & -3.00000000016300 & 0.00000000000242 \\ -0.99999999998137 & -4.00000000006629 & 0.99999999997043 \end{pmatrix},$$

$$X_2^{(17)} = \begin{pmatrix} 7.99999999987510 & -6.99999999990448 & -1.00000000002646 & -7.99999999997713 \\ -2.00000000000255 & 9.99999999998564 & 1.99999999995709 & 1.99999999993493 \\ -1.99999999977371 & 0.00000000021691 & 11.99999999995774 & 2.00000000004780 \\ -10.00000000005446 & -2.00000000009692 & 1.99999999992497 & 9.99999999984922 \end{pmatrix}.$$

We see that the algorithm computes a good approximation of the exact solution. In FIGURE 1, $\log_{10}\delta_k$ together with $\log_{10}\eta_k$ versus k are displayed, where

$$\eta_k = \frac{\|X^* - X^{(k)}\|}{\|X^*\|}.$$

Now, we consider Problem II corresponding to (4.1). Let $V = (V_1, V_2)$, where

$$V_1 = \begin{pmatrix} 2 & 1 & -3 \\ -1 & 3 & 2 \\ 3 & 1 & 2 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & -1 & -2 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

According the method described in Subsection 3.2, we obtain

$$\widehat{F}_1 = \frac{1}{90}\begin{pmatrix} 13815 & -17859 & 1922 \\ 6993 & -1068 & 6271 \\ -1899 & -14895 & -6426 \\ 16524 & -22329 & 2463 \\ 2160 & 861 & -6103 \end{pmatrix},$$

$$\widehat{F}_2 = \frac{1}{90}\begin{pmatrix} 2511 & -6001 & 7530 & -6466 \\ -7110 & 18083 & -579 & -1804 \\ -18477 & 9695 & -11964 & -17302 \\ 12312 & -328 & 5118 & -2764 \end{pmatrix}.$$

We apply Algorithm 1 to solve system (3.7). The algorithm converges in 17 iterations and provides the approximate solution $Z^{(17)} = (Z_1^{(17)}, Z_2^{(17)})$, where

$$Z_1^{(17)} = \begin{pmatrix} 2.44444444446260 & 1.44444444445492 & 9.22222222218051 \\ 7.61111111110360 & -5.88888888885177 & 0.05555555555165 \\ -3.11111111111684 & -6.11111111109703 & -0.55555555555053 \end{pmatrix},$$

$$Z_2^{(17)} = \begin{pmatrix} 6.60000000002732 & -7.10000000002355 & -0.89999999999066 & -8.40000000000365 \\ -1.50000000000030 & 11.40000000000025 & 1.80000000001034 & 1.40000000001344 \\ -4.10000000005157 & -1.80000000005236 & 10.60000000001025 & 0.99999999998851 \\ -9.59999999998482 & -2.39999999997847 & 2.40000000001808 & 9.40000000003345 \end{pmatrix}.$$

In this case, we obtain

$$\begin{aligned} \widetilde{X}_1^{(17)} &= W_1 + Z_1^{(17)} \\ &= \begin{pmatrix} 5.00000000001816 & 2.00000000001047 & 6.99999999995828 \\ 5.99999999999248 & -2.99999999996288 & -0.00000000000391 \\ -1.00000000000573 & -3.99999999998592 & 1.00000000000503 \end{pmatrix}, \\ \widetilde{X}_2^{(17)} &= W_2 + Z_2^{(17)} \\ &= \begin{pmatrix} 8.00000000002732 & -7.00000000002355 & -0.99999999999066 & -8.00000000000365 \\ -2.00000000000030 & 10.00000000000025 & 2.00000000001034 & 2.00000000001344 \\ -2.00000000005157 & -0.00000000005236 & 12.00000000001025 & 1.99999999998851 \\ -9.99999999998482 & -1.99999999997847 & 2.00000000001808 & 10.00000000003345 \end{pmatrix}. \end{aligned}$$

As we observe $\widetilde{X}_1^{(17)}$ and $\widetilde{X}_2^{(17)}$ are, respectively, in good agreement with X_1^* and X_2^*, which are generalized centro-symmetric matrices. In FIGURE 2, $\log_{10}\delta_k$ together with $\log_{10}\eta_k$ versus k are depicted, where δ_k is as before and

$$\eta_k = \frac{\|\widetilde{X}^* - \widetilde{X}^{(k)}\|}{\|\widetilde{X}^*\|}.$$

Example 4.2. In this example, we consider the coupled Sylvester-transpose matrix equations

$$\left\{ \begin{array}{l} \left(A_{11}X_1B_{11} + X_1^T\right) + \left(A_{12}X_2B_{12} + X_2^T\right) = F_1, \\ \left(X_1 + C_{21}X_1^TD_{21}\right) + \left(X_2 + C_{22}X_2^TD_{22}\right) = F_2, \end{array} \right. \tag{4.2}$$

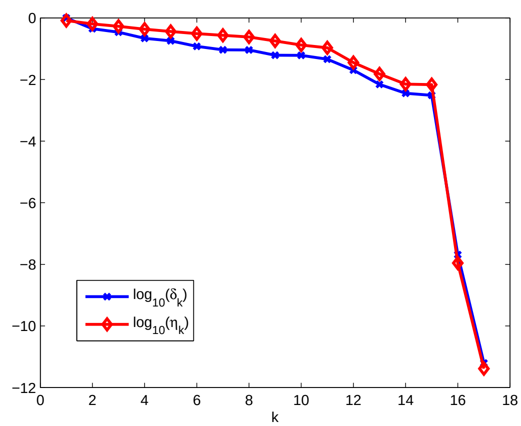

FIGURE 2. $\log_{10}\delta_k$ and $\log_{10}\eta_k$ versus k for Example 1 (Problem II).

where

$$A_{11}=\begin{pmatrix}1&2&1&2\\1&0&1&-1\\0&1&1&2\\0&1&1&0\end{pmatrix},\quad B_{11}=\begin{pmatrix}2&0&1&1\\1&-1&-1&-1\\2&-2&1&1\\2&0&1&2\end{pmatrix},$$

$$A_{12}=\begin{pmatrix}-1&2&1&-1\\0&1&4&1\\0&2&-2&2\\2&1&2&1\end{pmatrix},\quad B_{12}=\begin{pmatrix}3&2&1&2\\1&0&1&-1\\0&1&2&2\\-1&1&1&0\end{pmatrix},$$

$$C_{21}=\begin{pmatrix}2&1&1&2\\1&5&0&-5\\-1&2&2&2\\-1&1&1&3\end{pmatrix},\quad D_{21}=\begin{pmatrix}1&-1&1&1\\2&0&1&1\\-1&1&1&0\\0&1&1&2\end{pmatrix},$$

$$C_{22}=\begin{pmatrix}1&2&1&4\\2&1&1&-1\\-1&1&-1&2\\4&1&1&2\end{pmatrix},\quad D_{22}=\begin{pmatrix}-1&1&0&1\\1&5&2&-5\\-2&2&1&2\\-1&1&1&1\end{pmatrix},$$

and

$$F_1=\begin{pmatrix}67&-40&2&29\\-46&36&30&14\\88&-3&29&25\\33&18&40&23\end{pmatrix},\quad F_2=\begin{pmatrix}-13&140&75&34\\43&3&-9&-30\\61&52&76&57\\47&116&95&24\end{pmatrix}.$$

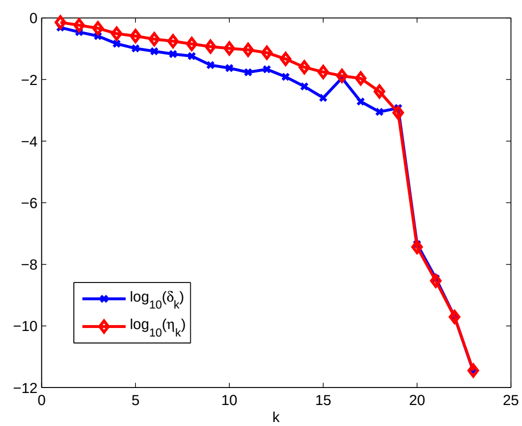

FIGURE 3. $\log_{10}\delta_k$ and $\log_{10}\eta_k$ versus k for Example 2 (Problem I).

System (4.2) is consistent over the generalized centro-symmetric matrices and has the exact solution $X^* = (X_1^*, X_2^*)$ with

$$X_1^* = \begin{pmatrix} -5 & 0 & 0 & 0 \\ 0 & 8 & 4 & 5 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 2 & 4 \end{pmatrix} \in \mathbb{CSR}_{R_1}^{4\times 4}, \quad X_2^* = \begin{pmatrix} 2 & 3 & -2 & 3 \\ 3 & 2 & -3 & 2 \\ -2 & -1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{pmatrix} \in \mathbb{CSR}_{R_2}^{4\times 4},$$

where

$$R_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{SOR}^{4\times 4}, \quad R_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \in \mathbb{SOR}^{4\times 4}.$$

By using Algorithm 1, we solve Problem I corresponding to system (4.2). All the assumptions are as the previous example. In this case, the method converges in 23 iterations and gives the approximate solution $X^{(23)} = (X_1^{(23)}, X_2^{(23)})$, where

$$X_1^{(23)} = \begin{pmatrix} -5.00000000000097 & 0 & 0 & 0 \\ 0 & 7.99999999999434 & 3.99999999998097 & 4.99999999996861 \\ 0 & -2.00000000001727 & 0.99999999999467 & 0.00000000001584 \\ 0 & 0.99999999997906 & 1.99999999998603 & 4.00000000000175 \end{pmatrix},$$

$$X_2^{(23)} = \begin{pmatrix} 1.99999999999156 & 3.00000000000913 & -1.99999999999557 & 3.00000000000045 \\ 3.00000000000913 & 1.99999999999156 & -3.00000000000045 & 1.99999999999557 \\ -2.00000000000422 & -0.99999999999960 & 4.00000000000418 & 2.00000000001128 \\ 0.99999999999960 & 2.00000000000422 & 2.00000000001128 & 4.00000000000418 \end{pmatrix}.$$

By comparing the obtained approximate solution $X^{(23)}$ and the exact solution, we observe that the proposed algorithm is quite suitable to solve system (4.2). For more investigation the graph of $\log_{10}\delta_k$ and $\log_{10}\eta_k$ versus k are shown in FIGURE 3.

Now, we consider Problem II corresponding to (4.2). Let $V=(V_1,V_2)$, where

$$V_1=\begin{pmatrix} 1 & 2 & 1 & 1\\ -1 & 1 & 2 & 1\\ 2 & 3 & 1 & 1\\ 1 & 2 & 3 & 1\end{pmatrix},\quad V_2=\begin{pmatrix} 2 & 1 & -1 & 1\\ 2 & -3 & 1 & 0\\ 0 & 3 & 2 & 1\\ 1 & 0 & 2 & -1\end{pmatrix}.$$

By the method mentioned in Subsection 3.2, we get

$$\widehat{F}_1=\frac{1}{2}\begin{pmatrix} 27 & -42 & -20 & 15\\ -89 & 42 & 20 & 19\\ 107 & 36 & 43 & 5\\ 40 & 44 & 51 & 42\end{pmatrix},\quad \widehat{F}_2=\frac{1}{2}\begin{pmatrix} -1 & 187 & 79 & 8\\ 95 & -41 & -60 & -63\\ 134 & 40 & 87 & 17\\ 88 & 118 & 109 & 55\end{pmatrix}.$$

Algorithm 1 to solve system (3.7) converges in 22 iterations and gives the the approximate solution $Z^{(22)}=(Z_1^{(22)},Z_2^{(22)})$, where

$$Z_1^{(22)}=\begin{pmatrix} -5.99999999997951 & 0 & 0 & 0\\ 0 & 7.00000000015436 & 2.00000000000119 & 3.99999999984046\\ 0 & -4.99999999998646 & 0.00000000001920 & -0.99999999995247\\ 0 & -0.99999999984194 & -0.99999999998000 & 2.99999999990840\end{pmatrix},$$

$$Z_2^{(22)}=\begin{pmatrix} 2.50000000001917 & 1.50000000000768 & -1.49999999998730 & 3.00000000004049\\ 1.50000000000768 & 2.50000000001917 & -3.00000000004049 & 1.49999999998730\\ -1.99999999994624 & -1.99999999999403 & 3.50000000006985 & 0.50000000007078\\ 1.99999999999403 & 1.99999999994624 & 0.50000000007078 & 3.50000000006985\end{pmatrix}.$$

In this case, we obtain

$$\begin{aligned}\widetilde{X}_1^{(22)} &= W_1+Z_1^{(22)}\\ &=\begin{pmatrix} -4.99999999997951 & 0 & 0 & 0\\ 0 & 8.00000000015436 & 4.00000000000119 & 4.99999999984046\\ 0 & -1.99999999998646 & 1.00000000001920 & 0.00000000004753\\ 0 & 1.00000000015806 & 2.00000000002000 & 3.99999999990840\end{pmatrix},\\ \widetilde{X}_2^{(22)} &= W_2+Z_2^{(22)}\\ &=\begin{pmatrix} 2.00000000001917 & 3.00000000000768 & -1.99999999998730 & 3.00000000004049\\ 3.00000000000768 & 2.00000000001917 & -3.00000000004049 & 1.99999999998730\\ -1.99999999994624 & -0.99999999999403 & 4.00000000006985 & 2.00000000007078\\ 0.99999999999403 & 1.99999999994624 & 2.00000000007078 & 4.00000000006985\end{pmatrix}.\end{aligned}$$

As seen, $\widetilde{X}_1^{(22)}$ and $\widetilde{X}_2^{(22)}$ are in good agreement with X_1^* and X_2^*, respectively, which are generalized centro-symmetric matrices. In Figure 4, $\log_{10}\delta_k$ and $\log_{10}\eta_k$ versus k are plotted, where δ_k and η_k are defined as the previous example.

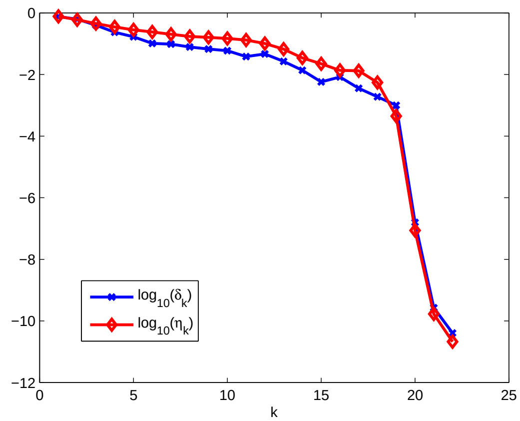

FIGURE 4. $\log_{10}\delta_k$ and $\log_{10}\eta_k$ versus k for Example 2 (Problem II).

5. Conclusion

In this paper, we have considered two main problems. The first problem, Problem I, concerned with solving the coupled Sylvester-transpose matrix equations over the generalized centro-symmetric matrices. For solving Problem I, an iterative algorithm (Algorithm 1) has been proposed which can automatically determine the solvability of the problem . When Problem I is consistent, for any initial generalized centro-symmetric solution group, Algorithm 1 computes the solution of Problem I within finite number of iteration steps in the absence of roundoff errors. It has been shown that the least-norm generalized centro-symmetric solution group of the coupled Sylvester-transpose matrix equations can be calculated by choosing a suitable initial generalized centro-symmetric solution group. The main aim of the second problem, Problem II, was to calculate the optimal approximate generalized centro-symmetric matrix group to an arbitrary matrix group. An approach has been given for solving Problem II. Finally, two numerical examples have been presented to demonstrate the validity of the theoretical results and to examine the applicability of Algorithm 1 for solving Problems I and II.

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