GRADIENT BASED ITERATIVE ALGORITHM FOR SOLVING THE GENERALIZED COUPLED SYLVESTER-TRANSPOSE AND CONJUGATE MATRIX EQUATIONS OVER REFLEXIVE (ANTI-REFLEXIVE) MATRICES

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Abstract. Linear matrix equations play an important role in many areas, such as control theory, system theory, stability theory and some other fields of pure and applied mathematics. In the present paper, we consider the generalized coupled Sylvester-transpose and conjugate matrix equations

$$\mathcal{T}_\nu(X) = F_\nu, \qquad \nu = 1, 2, \ldots, N,$$

where $X = (X_1, X_2, \ldots, X_p)$ is a group of unknown matrices and for $\nu = 1, 2, \ldots, N$,

$$\mathcal{T}_\nu(X) = \sum_{i=1}^{p}\sum_{\mu=1}^{s_1} A_{\nu i\mu} X_i B_{\nu i\mu} + \sum_{\mu=1}^{s_2} C_{\nu i\mu} X_i^T D_{\nu i\mu} + \sum_{\mu=1}^{s_3} M_{\nu i\mu} \overline{X}_i N_{\nu i\mu} + \sum_{\mu=1}^{s_4} H_{\nu i\mu} X_i^H G_{\nu i\mu},$$

in which $A_{\nu i\mu}, B_{\nu i\mu}, C_{\nu i\mu}, D_{\nu i\mu}, M_{\nu i\mu}, N_{\nu i\mu}, H_{\nu i\mu}, G_{\nu i\mu}$ and F_ν are given matrices with suitable dimensions defined over complex number field. By using the hierarchical identification principle, an iterative algorithms is proposed for solving the above coupled linear matrix equations over the group of reflexive (anti-reflexive) matrices. Meanwhile, sufficient conditions are established which guarantee the convergence of the presented algorithm. Finally, some numerical examples are given to demonstrate the validity of our theoretical results and the efficiency of the algorithm for solving the mentioned coupled linear matrix equations.

Keywords: Generalized Sylvester-transpose and conjugate matrix equation, Iterative algorithm, Reflexive (anti-reflexive) matrix.
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1. Introduction

In this paper, the following notations are utilized. We use $\text{tr}(A)$, A^T, A^H, $\overline{A}$ to denote the trace, the transpose, the conjugate transposed, the conjugate of the matrix A, respectively. Moreover, $\mathbb{C}^{m\times n}$ represents the set of all $m \times n$ complex matrices. The set of all symmetric

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orthogonal matrices, also known as reflection matrices, in $\mathbb{C}^{n\times n}$ is represented by $\mathbb{SOC}^{n\times n}$, i.e., $P \in \mathbb{SOC}^{n\times n}$ if and only if $P = P^H = P^{-1}$.

Definition 1.1. [6] *Consider two arbitrary given matrices $P, Q \in \mathbb{SOC}^{n\times n}$. The matrix $A \in \mathbb{C}^{n\times n}$ is called a reflexive matrix, with respect to P and Q, if $A = PAQ$. The set of all $n \times n$ reflexive, (P,Q)-reflexive, matrices is denoted by $\mathbb{C}_r^{n\times n}(P,Q)$.*

Definition 1.2. [6] *Consider two arbitrary given matrices $P, Q \in \mathbb{SOC}^{n\times n}$. The matrix $A \in \mathbb{C}^{n\times n}$ is called an anti-reflexive matrix, with respect to P and Q, if $A = -PAQ$. The set of all $n \times n$ anti-reflexive, (P,Q)-anti-reflexive, matrices is denoted by $\mathbb{C}_a^{n\times n}(P,Q)$.*

In the literature, the problem of finding solutions of several linear matrix equations has been investigated widely, for more details see [1, 3, 4, 12, 13, 15, 16, 17, 21, 22, 24, 26, 28] and the references therein. Before stating the main problems of this paper, we briefly review some of the works which have been recently presented in the field of linear matrix equations.

Recently, the idea of conjugate gradient (CG) method has been developed for constructing iterative algorithms to compute the solutions of different kinds of linear matrix equations over reflexive and anti-reflexive, generalized bisymmetric, generalized centro-symmetric, mirror-symmetric, skew-symmetric and (P,Q)-reflexive matrices, for more details see [7, 8, 9, 14, 15, 17, 18, 19, 23, 25].

Gradient based iterative algorithm is a different common approach for solving linear matrix equations. For instance, Ding et al. [13] have considered the solution of $AXB = F$ and $AXB + CXD = F$. Dehghan and Hajarian [10] proposed two algorithms for solving the following the matrix equation

$$\sum_{i=1}^{p} A_i X B_i + \sum_{j=1}^{q} C_j Y D_j = F, \tag{1.1}$$

over reflexive and anti-reflexive matrices.

In [11], in fact, the authors have employed the idea of the Gradient based iterative method to construct two iterative algorithms for computing the generalized bisymmetric and skew-symmetric solutions of the linear matrix equation

$$\sum_{i=1}^{l} A_i Y B_i = C. \tag{1.2}$$

Moreover, Song et al. [22] have considered the following coupled Sylvester-transpose matrix equations

$$\sum_{\eta=1}^{p} \left(A_{i\eta} X_\eta B_{i\eta} + C_{i\eta} X_\eta^T D_{i\eta} \right) = F_i, \qquad i = 1, 2, \ldots, N, \tag{1.3}$$

where $A_{i\eta} \in \mathbb{R}^{m_i \times l_\eta}$, $B_{i\eta} \in \mathbb{R}^{n_\eta \times p_i}$, $C_{i\eta} \in \mathbb{R}^{m_i \times n_\eta}$, $D_{i\eta} \in \mathbb{R}^{l_\eta \times p_i}$, $F_i \in \mathbb{R}^{m_i \times p_i}$, $i = 1, \ldots, N$, $\eta = 1, \ldots, p$, are given matrices and $X_\eta \in \mathbb{R}^{l_\eta \times n_\eta}, \eta \in I[1,p]$ are the matrices to be determined.

For simplicity, we use the following linear operator

$$\mathcal{T}_\nu : \mathbb{C}^{p_1 \times p_1} \times \cdots \times \mathbb{C}^{p_p \times p_p} \to \mathbb{C}^{\lambda_\nu \times \gamma_\nu}$$

such that

$$\mathcal{T}_\nu(X) = \sum_{i=1}^{p} \left(\sum_{\mu=1}^{s_1} A_{\nu i \mu} X_i B_{\nu i \mu} + \sum_{\mu=1}^{s_2} C_{\nu i \mu} X_i^T D_{\nu i \mu} + \sum_{\mu=1}^{s_3} M_{\nu i \mu} \overline{X}_i N_{\nu i \mu} + \sum_{\mu=1}^{s_4} H_{\nu i \mu} X_i^H G_{\nu i \mu} \right),$$

where $X = (X_1, X_2, ..., X_p)$ and $\nu = 1, 2, \ldots, N$.

In the present work, we will construct a gradient based algorithm for solving the following coupled linear matrix equations

$$\mathcal{T}_\nu(X) = F_\nu, \qquad \nu \in I[1, N], \tag{1.4}$$

over the group of reflexive (anti-reflexive) matrices. Evidently, the coupled linear matrix equations (1.4) are extremely general and include many investigated linear matrix equations such as the generalized (coupled) Sylvester and Lyapunov matrix equations, Eqs (1.1), (1.2) and (1.3).

1.1. **Problem reformulation.** We will focus on the following problems and construct an iterative algorithm for obtaining the solutions of these problems.

Problem 1. Assume that the matrices $A_{\nu i\mu} \in \mathbb{C}^{\lambda_\nu \times p_i}, B_{\nu i\mu} \in \mathbb{C}^{p_i \times \gamma_\nu}$, $\mu \in I[1, s_1]$, $C_{\nu i\mu} \in \mathbb{C}^{\lambda_\nu \times p_i}, D_{\nu i\mu} \in \mathbb{C}^{p_i \times \gamma_\nu}$, $\mu \in I[1, s_2]$, $M_{\nu i\mu} \in \mathbb{C}^{\lambda_\nu \times p_i}, N_{\nu i\mu} \in \mathbb{C}^{p_i \times \gamma_\nu}$, $\mu \in I[1, s_3]$, $H_{\nu i\mu} \in \mathbb{C}^{\lambda_\nu \times p_i}, G_{\nu i\mu} \in \mathbb{C}^{p_i \times \gamma_\nu}$, $\mu \in I[1, s_4]$, $F_\nu \in \mathbb{C}^{\lambda_\nu \times \gamma_\nu}$ and $R_i, Q_i \in \mathbb{SOC}^{p_i \times p_i}$ are given. Find the reflexive matrix group $(X_1, X_2, \ldots, X_p)$ such that $X_i \in \mathbb{C}_r^{p_i \times p_i}(R_i, Q_i)$ and satisfy (1.4) where $i \in I[1, p], \nu \in I[1, N]$.

Problem 2. Assume that the matrices $A_{\nu i\mu} \in \mathbb{C}^{\lambda_\nu \times p_i}, B_{\nu i\mu} \in \mathbb{C}^{p_i \times \gamma_\nu}$, $\mu \in I[1, s_1]$, $C_{\nu i\mu} \in \mathbb{C}^{\lambda_\nu \times p_i}, D_{\nu i\mu} \in \mathbb{C}^{p_i \times \gamma_\nu}$, $\mu \in I[1, s_2]$, $M_{\nu i\mu} \in \mathbb{C}^{\lambda_\nu \times p_i}, N_{\nu i\mu} \in \mathbb{C}^{p_i \times \gamma_\nu}$, $\mu \in I[1, s_3]$, $H_{\nu i\mu} \in \mathbb{C}^{\lambda_\nu \times p_i}, G_{\nu i\mu} \in \mathbb{C}^{p_i \times \gamma_\nu}$, $\mu \in I[1, s_4]$, $F_\nu \in \mathbb{C}^{\lambda_\nu \times \gamma_\nu}$ and $R_i, Q_i \in \mathbb{SOC}^{p_i \times p_i}$ are given. Find the anti-reflexive matrix group $(X_1, X_2, \ldots, X_p)$ such that $X_i \in \mathbb{C}_a^{p_i \times p_i}(R_i, Q_i)$ and satisfy (1.4) where $i \in I[1, p], \nu \in I[1, N]$.

The reminder of this paper is organized as follows. In Section 2, we recall some definitions and theorems which are used for presenting our main theoretical results. In Section 3, first, we investigate the solvability of Problems 1 and 2. Then, an algorithm is proposed for solving these problems. Convergence analysis of the algorithm is also discussed. In order to illustrate the validity and applicability of our presented results, we give some numerical examples in Section 4. Finally, the paper is ended with a brief conclusion in Section 5.

2. Preliminaries

In this section, we review some necessary principles and definitions which are utilized throughout this work.

Definition 2.1. *Suppose that $Y = (Y_1, Y_2, \ldots, Y_k)$ and $Z = (Z_1, Z_2, \ldots, Z_k)$ where $Y_i, Z_i \in \mathbb{C}^{r_i \times s_i}$ for $i = 1, 2, \ldots, k$. We define the inner product $< ., . >$ as follows:*

$$< Y, Z >= \mathrm{Re}[\sum_{i=1}^{k} \mathrm{tr}(Y_j^H Z_j)]. \tag{2.1}$$

Remark 2.2. *For $Y = (Y_1, Y_2, \ldots, Y_k)$, $Y_i \in \mathbb{C}^{r_i \times s_i}$ for $i \in I[1, k]$, the norm of Y is defined by $\|Y\|^2 = \mathrm{Re}[\sum_{i=1}^{k} \mathrm{tr}(Y_i^H Y_i)]$.*

Assume that $A = [a_{ij}]_{m\times s}$ and $B = [b_{ij}]_{n\times q}$ defined over complex (real) number field, the Kronecker product of the matrices A and B is defined as the $mn\times sq$ matrix $A\otimes B = [a_{ij}B]$. The "vec" operator transforms a matrix A of size $m \times s$ to a vector $a = vec(A)$ of size $ms \times 1$ by stacking the columns of A. In this paper, the following relation is utilized (See [2])

$$\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X).$$

Lemma 2.3. *Let $X \in \mathbb{R}^{m\times n}$ be an arbitrary matrix. Then*

$$vec(X^T) = P(m, n)vec(X),$$

where $P(m, n)$ is uniquely determined by the integers m and n.

Proof. See [6]. □

Remark 2.4. *Let $X \in \mathbb{C}^{m\times n}$ be an arbitrary matrix. Then*

$$\text{vec}(X^T) = P(m, n)\text{vec}(X),$$

and

$$\text{vec}(X^H) = \text{vec}(\overline{X}^T) = P(m, n)\text{vec}(\overline{X}),$$

where $P(m, n)$ is uniquely determined by the integers m and n.

Some properties of the matrix $P(m, n)$ are given as follows ([6, 17, 27]):

1. For two arbitrary integers m and n, $P(m, n)$ has the following explicit form

$$P(m,n) = \begin{pmatrix} E_{11}^T & E_{12}^T & \cdots & E_{1n}^T \\ E_{21}^T & E_{22}^T & \cdots & E_{2n}^T \\ \vdots & \vdots & \ddots & \vdots \\ E_{m1}^T & E_{m2}^T & \cdots & E_{mn}^T \end{pmatrix}_{mn\times mn}.$$

where each E_{ij} for $i \in I[1, m]$ and $j \in I[1, n]$, is an $m \times n$ matrix with the element at position (i, j) being one and the others being zero.

2. For two arbitrary integers m and n, $P(m, n)$ is the unitary matrix, i.e., $P(m, n)P^T(m, n) = P^T(m, n)P(m, n) = I_{mn}$.
3. For two arbitrary integers m and n, $P(m, n) = P^T(n, m)$.

3. Main results

This section consists of two parts. In the first part, the solvability of Problems 1 and 2 is discussed. In the second part, we give an iterative algorithm for solving the coupled linear matrix equations (1.4) over reflexive (anti-reflexive) matrices. In addition, sufficient conditions are established which guarantee the convergence of the proposed algorithm.

3.1. Solvability of the Problems 1 and 2.

By some straightforward computations, we can prove the following lemmas.

Lemma 3.1. *Problem 1 is solvable if and only if the system of matrix equations*

$$(\mathcal{T}_\nu(X), \mathcal{T}_\nu(Z)) = (F_\nu, F_\nu), \qquad \nu \in I[1, N], \tag{3.1}$$

are consistent, where $Z = (Z_1, Z_2, \ldots, Z_p)$ is defined such that $Z_i = R_iX_iQ_i$ in which the matrices $R_i, Q_i \in \mathbb{SOC}^{p_i\times p_i}$ are given for $i \in I[1, p]$.

Proof. Without loss of generality, we may assume that $s_1 = s_2 = s_3 = s_4 = s$. Suppose that the matrix group $X = (X_1, X_2, \ldots, X_p)$ satisfies in Eq. (3.1). Therefore,

$$\begin{cases} \sum_{i=1}^{p}\sum_{\mu=1}^{s} A_{\nu i\mu}X_iB_{\nu i\mu} + C_{\nu i\mu}X_i^TD_{\nu i\mu} + M_{\nu i\mu}\overline{X}_iN_{\nu i\mu} + H_{\nu i\mu}X_i^HG_{\nu i\mu} = F_\nu \\ \sum_{i=1}^{p}\sum_{\mu=1}^{s} A_{\nu i\mu}R_iX_iQ_iB_{\nu i\mu} + C_{\nu i\mu}Q_i^TX_i^TR_i^TD_{\nu i\mu} + M_{\nu i\mu}\overline{R}_i\overline{X}_i\overline{Q}_iN_{\nu i\mu} + H_{\nu i\mu}Q_iX_i^HR_iG_{\nu i\mu} = F_\nu. \end{cases}$$

Now, we define the matrix group $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_p)$ such that

$$\widetilde{X}_i = \frac{X_i + R_iX_iQ_i}{2}, \qquad i = 1, 2, ..., p.$$

Evidently, $\widetilde{X}$ is a group of reflexive matrices, i.e., $\widetilde{X}_i \in \mathbb{C}_r^{p_i\times p_i}(R_i, Q_i)$, for $i = 1, 2, ..., p$. It is easy to verify that for $\nu \in I[1, N]$

$$\sum_{i=1}^{p}\sum_{\mu=1}^{s} A_{\nu i\mu}\widetilde{X}_iB_{\nu i\mu} + C_{\nu i\mu}\widetilde{X}_i^TD_{\nu i\mu} + M_{\nu i\mu}\overline{\widetilde{X}}_iN_{\nu i\mu} + H_{\nu i\mu}\widetilde{X}_i^HG_{\nu i\mu} = F_\nu,$$

which shows that the matrix groups $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_p)$ is the solution of Problem 1.

Conversely, let the matrix group $X = (X_1, X_2, \ldots, X_p)$ be the solution of Problem 1. That is

$$X_i \in \mathbb{C}_r^{p_i\times p_i}(R_i, Q_i), \qquad i = 1, 2, ..., p,$$

and $X = (X_1, X_2, \ldots, X_p)$ satisfy in Eq. (1.4). Hence, the proof can be concluded immediately from the fact that $X_i = R_iX_iQ_i$ for $i = 1, 2, ..., p$. □

Analogous to the strategy employed in the proof of Lemma 3.1, we may establish the following lemma.

Lemma 3.2. *Problem 2 is solvable if and only if the system of matrix equations*

$$(\mathcal{T}_\nu(X), \mathcal{T}_\nu(Z)) = (F_\nu, -F_\nu), \qquad \nu \in I[1, N]. \tag{3.2}$$

are consistent, where $Z = (Z_1, Z_2, \ldots, Z_p)$ is defined such that $Z_i = R_iX_iQ_i$ in which the matrices $R_i, Q_i \in \mathbb{SOC}^{p_i\times p_i}$ are given for $i \in I[1, p]$.

Assume that the matrices ψ_1, ψ_2, ψ_3 and ψ_4 are defined as follows:

$$\psi_1 = \begin{pmatrix} \sum_{\mu=1}^{s_1} B_{11\mu}^T \otimes A_{11\mu} & \cdots & \sum_{\mu=1}^{s_1} B_{1p\mu}^T \otimes A_{1p\mu} \\ \vdots & & \vdots \\ \sum_{\mu=1}^{s_1} B_{N1\mu}^T \otimes A_{N1\mu} & \cdots & \sum_{\mu=1}^{s_1} B_{Np\mu}^T \otimes A_{Np\mu} \\ \sum_{\mu=1}^{s_1} B_{11\mu}^TQ_1^T \otimes A_{11\mu}R_1 & \cdots & \sum_{\mu=1}^{s_1} B_{1p\mu}^TQ_p^T \otimes A_{1p\mu}R_p \\ \vdots & & \vdots \\ \sum_{\mu=1}^{s_1} B_{N1\mu}^TQ_1^T \otimes A_{N1\mu}R_1 & \cdots & \sum_{\mu=1}^{s_1} B_{Np\mu}^TQ_p^T \otimes A_{Np\mu}R_p \end{pmatrix},$$

$$\psi_2 = \begin{pmatrix} \sum_{\mu=1}^{s_2}(D_{11\mu}^T \otimes C_{11\mu})P(p_1,p_1) & \cdots & \sum_{\mu=1}^{s_2}(D_{1p\mu}^T \otimes C_{1p\mu})P(p_p,p_p) \\ \vdots & & \vdots \\ \sum_{\mu=1}^{s_2}(D_{N1\mu}^T \otimes C_{N1\mu})P(p_1,p_1) & \cdots & \sum_{\mu=1}^{s_2}(D_{Np\mu}^T \otimes C_{Np\mu})P(p_p,p_p) \\ \sum_{\mu=1}^{s_2}(D_{11\mu}^T R_1 \otimes C_{11\mu}Q_1^T)P(p_1,p_1) & \cdots & \sum_{\mu=1}^{s_2}(D_{1p\mu}^T R_p \otimes C_{1p\mu}Q_p^T)P(p_p,p_p) \\ \vdots & & \vdots \\ \sum_{\mu=1}^{s_2}(D_{N1\mu}^T R_1 \otimes C_{N1\mu}Q_1^T)P(p_1,p_1) & \cdots & \sum_{\mu=1}^{s_2}(D_{Np\mu}^T R_p \otimes C_{Np\mu}Q_p^T)P(p_p,p_p) \end{pmatrix},$$

$$\psi_3 = \begin{pmatrix} \sum_{\mu=1}^{s_3} N_{11\mu}^T \otimes M_{11\mu} & \cdots & \sum_{\mu=1}^{s_3} N_{1p\mu}^T \otimes M_{1p\mu} \\ \vdots & & \vdots \\ \sum_{\mu=1}^{s_3} N_{N1\mu}^T \otimes M_{N1\mu} & \cdots & \sum_{\mu=1}^{s_3} N_{Np\mu}^T \otimes M_{Np\mu} \\ \sum_{\mu=1}^{s_3} N_{11\mu}^T Q_1 \otimes M_{11\mu}\overline{R}_1 & \cdots & \sum_{\mu=1}^{s_3} N_{1p\mu}^T Q_p \otimes M_{1p\mu}\overline{R}_p \\ \vdots & & \vdots \\ \sum_{\mu=1}^{s_3} N_{N1\mu}^T Q_1 \otimes M_{N1\mu}\overline{R}_1 & \cdots & \sum_{\mu=1}^{s_3} N_{Np\mu}^T Q_p \otimes M_{Np\mu}\overline{R}_p \end{pmatrix},$$

and

$$\psi_4 = \begin{pmatrix} \sum_{\mu=1}^{s_4}(G_{11\mu}^T \otimes H_{11\mu})P(p_1,p_1) & \cdots & \sum_{\mu=1}^{s_4}(G_{1p\mu}^T \otimes H_{1p\mu})P(p_p,p_p) \\ \vdots & & \vdots \\ \sum_{\mu=1}^{s_4}(G_{N1\mu}^T \otimes H_{N1\mu})P(p_1,p_1) & \cdots & \sum_{\mu=1}^{s_4}(G_{Np\mu}^T \otimes H_{Np\mu})P(p_p,p_p) \\ \sum_{\mu=1}^{s_4}(G_{11\mu}^T R_1^T \otimes H_{11\mu}Q_1)P(p_1,p_1) & \cdots & \sum_{\mu=1}^{s_4}(G_{1p\mu}^T R_p^T \otimes H_{1p\mu}Q_p)P(p_p,p_p) \\ \vdots & & \vdots \\ \sum_{\mu=1}^{s_4}(G_{N1\mu}^T R_1^T \otimes H_{N1\mu}Q_1)P(p_1,p_1) & \cdots & \sum_{\mu=1}^{s_4}(G_{Np\mu}^T R_p^T \otimes H_{Np\mu}Q_p)P(p_p,p_p) \end{pmatrix}.$$

Let $\Pi = (\Psi_1 + \Psi_2, \Psi_3 + \Psi_4)$,

$$\upsilon = \begin{pmatrix} \mathrm{vec}(X_1) \\ \vdots \\ \mathrm{vec}(X_p) \\ \mathrm{vec}(\overline{X}_1) \\ \vdots \\ \mathrm{vec}(\overline{X}_p) \end{pmatrix}, \quad \Omega_r = \begin{pmatrix} \mathrm{vec}(F_1) \\ \vdots \\ \mathrm{vec}(F_N) \\ \mathrm{vec}(F_1) \\ \vdots \\ \mathrm{vec}(F_N) \end{pmatrix}, \quad \text{and} \quad \Omega_a = \begin{pmatrix} \mathrm{vec}(F_1) \\ \vdots \\ \mathrm{vec}(F_N) \\ -\mathrm{vec}(F_1) \\ \vdots \\ -\mathrm{vec}(F_N) \end{pmatrix}.$$

It is not difficult to see that Eqs. (3.1) and (3.2) are equivalent to the following to the following linear systems, respectively,

$$\Pi\upsilon = \Omega_r,$$

and

$$\Pi \upsilon = \Omega_a.$$

Now, we can conclude the following theorems from Lemmas 3.1 and 3.2.

Theorem 3.3. *Problem 1 has a unique solution if and only if $Rank(\Pi, \Omega_r) = Rank(\Pi)$ and Π has full column rank. The unique reflexive solution group of Problem 1 can be calculated as follows:*

$$X_i = \frac{\widetilde{X}_i + R_i \widetilde{X}_i Q_i}{2}, \quad i = 1, 2, \ldots, p,$$

where $\widetilde{X} = (\widetilde{X}_1, \ldots, \widetilde{X}_p, \overline{\widetilde{X}}_1, \ldots, \overline{\widetilde{X}}_p)$ is derived such that

$$\mathrm{vec}(\widetilde{X}) = (\Pi^H \Pi)^{-1} \Pi^H \Omega_r.$$

Theorem 3.4. *Problem 2 has a unique solution if and only if $Rank(\Pi, \Omega_a) = Rank(\Pi)$ and Π has full column rank. The unique anti-reflexive solution group of Problem 2 can be calculated as follows:*

$$X_i = \frac{\widetilde{X}_i - R_i \widetilde{X}_i Q_i}{2}, \quad i = 1, 2, \ldots, p,$$

where $\widetilde{X} = (\widetilde{X}_1, \ldots, \widetilde{X}_p, \overline{\widetilde{X}}_1, \ldots, \overline{\widetilde{X}}_p)$ is derived such that

$$\mathrm{vec}(\widetilde{X}) = (\Pi^H \Pi)^{-1} \Pi^H \Omega_a.$$

3.2. **Proposed algorithms.** In this section, by developing the idea of gradient based iterative method, we propose an iterative algorithm to compute the reflexive (anti-reflexive) solution group of Eq. (1.4).

Before presenting the algorithm, for $j = 1, 2, 3, 4$, we define the linear operators $\mathcal{S}_j(V)$ as follows:

$$\mathcal{S}_j : \mathbb{C}^{\lambda_1 \times \gamma_1} \times \cdots \times \mathbb{C}^{\lambda_N \times \gamma_N} \to \mathbb{C}^{p_1 \times p_1} \times \cdots \times \mathbb{C}^{p_p \times p_p}$$
$$V \to \mathcal{S}_j(V),$$

such that $V = (V_1, V_2, \ldots, V_N)$, $\mathcal{S}_j(V) = (S_{1j}(V), S_{2j}(V), \ldots, S_{pj}(V))$ and for $i \in I[1, p]$

$$S_{i1}(V) = \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_1} A_{\nu i\mu}^H V_\nu B_{\nu i\mu}^H, \qquad S_{i2}(V) = \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_2} \overline{D_{\nu i\mu}} V_\nu^T \overline{C}_{\nu i\mu},$$

$$S_{i3}(V) = \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_3} \overline{M_{\nu i\mu}}^H \overline{V}_\nu \overline{N}_{\nu i\mu}^H, \qquad S_{i4}(V) = \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_4} G_{\nu i\mu} V_\nu^H H_{\nu i\mu}.$$

Algorithm 1. (`Proposed algorithm for Problems` 1 `and` 2)

Step 1: `Input the matrices` $A_{\nu i\mu} \in \mathbb{C}^{\lambda_\nu \times p_i}, B_{\nu i\mu} \in \mathbb{C}^{p_i \times \gamma_\nu}$, $\mu \in I[1, s_1]$, $C_{\nu i\mu} \in \mathbb{C}^{\lambda_\nu \times p_i}, D_{\nu i\mu} \in \mathbb{C}^{p_i \times \gamma_\nu}$, $\mu \in I[1, s_2]$, $M_{\nu i\mu} \in \mathbb{C}^{\lambda_\nu \times p_i}, N_{\nu i\mu} \in \mathbb{C}^{p_i \times \gamma_\nu}$, $\mu \in I[1, s_3]$, $H_{\nu i\mu} \in \mathbb{C}^{\lambda_\nu \times p_i}, G_{\nu i\mu} \in \mathbb{C}^{p_i \times \gamma_\nu}$, $\mu \in I[1, s_4]$, $F_\nu \in \mathbb{C}^{\lambda_\nu \times \gamma_\nu}$, $R_i, Q_i \in \mathbb{SOC}^{p_i \times p_i}$, for $i \in I[1, p]$, and $\nu \in I[1, N]$.

Step 2: `Set` $\kappa = 0$ `for computing the solution of Problem` 1, `and` $\kappa = 1$ `for Problem` 2. `Choose a tolerance` ε .

Step 3: `If` $\kappa = 0$, `select the initial reflexive group of matrices`

$$X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \ldots, X_p^{(1)}).$$

`Otherwise select the initial anti-reflexive group of matrices`

$$X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \ldots, X_p^{(1)}).$$

Step 4: Set $t = 1$. For $\nu = 1, 2, 3, \ldots, N$, compute $R_\nu^{(1)} = F_\nu - \mathcal{T}_\nu(X^{(1)})$.

Step 5: Choose the integer values α_{ij}, for $i \in I[1, p]$ and $j = 1, 2, 3, 4$, such that if $S_{ij}(V) \equiv 0$ for all $V = (V_1, V_2, \ldots, V_N)$ where $V_\nu \in \mathbb{C}^{\lambda_\nu \times \gamma_\nu}$ for $\nu \in I[1, N]$; Set $\alpha_{ij} = 0$; Otherwise set $\alpha_{ij} = 1$.

Step 6: For $i = 1, 2, \ldots, p$, Do:

If $\alpha_{i1} = 1$, set

$$U_i^{(t+1)} = X_i^{(t)} + \frac{\omega}{2Ns_1}(S_{i1}(R^{(t)}) + (-1)^\kappa R_i S_{i1}(R^{(t)})Q_i);$$

Otherwise $U_i^{(t+1)} = 0$.

If $\alpha_{i2} = 1$, set

$$Y_i^{(t+1)} = X_i^{(t)} + \frac{\omega}{2Ns_2}(S_{i2}(R^{(t)}) + (-1)^\kappa R_i S_{i2}(R^{(t)})Q_i);$$

Otherwise $Y_i^{(t+1)} = 0$.

If $\alpha_{i3} = 1$, set

$$Z_i^{(t+1)} = X_i^{(t)} + \frac{\omega}{2Ns_3}(S_{i3}(R^{(t)}) + (-1)^\kappa R_i S_{i3}(R^{(t)})Q_i);$$

Otherwise $Z_i^{(t+1)} = 0$.

If $\alpha_{i4} = 1$, set

$$W_i^{(t+1)} = X_i^{(t)} + \frac{\omega}{2Ns_4}(S_{i4}(R^{(t)}) + (-1)^\kappa R_i S_{i4}(R^{(t)})Q_i);$$

Otherwise $W_i^{(t+1)} = 0$.

Step 7: For $i = 1, 2, \ldots, p$, calculate $\beta_i := \alpha_{i1} + \alpha_{i2} + \alpha_{i3} + \alpha_{i4}$; Set

$$X_i^{(t+1)} = \frac{1}{\beta_i}(U_i^{(t+1)} + Y_i^{(t+1)} + Z_i^{(t+1)} + W_i^{(t+1)}).$$

Step 8: Compute $R^{(t+1)} = (R_1^{(t+1)}, R_2^{(t+1)}, \ldots, R_N^{(t+1)})$ where

$$R_\nu^{(t+1)} = F_\nu - \mathcal{T}_\nu(X^{(t+1)}), \qquad \nu = 1, 2, \ldots, N.$$

Step 9: If $\|R^{(t+1)}\| < \varepsilon$ then Stop; else, set $t = t + 1$ and go to Step 6.

Theorem 3.5. *Suppose that the linear coupled matrix equations* (1.4) *has unique reflexive solution group* $X^* = (X_1^*, X_2^*, \ldots, X_p^*)$. *If the parameter* ω *satisfies the inequality*

$$0 < \omega < \frac{2}{\Theta}, \tag{3.3}$$

such that $\Theta = \sum_{\nu=1}^{N} \sum_{i=1}^{p} \Theta_{i\nu}$ *where,*

$$\Theta_{1\nu} = \sum_{\mu=1}^{s_1} \|A_{\nu i\mu}\|^2 \|B_{\nu i\mu}\|^2, \qquad \Theta_{2\nu} = \sum_{\mu=1}^{s_2} \|C_{\nu i\mu}\|^2 \|D_{\nu i\mu}\|^2$$
$$\Theta_{3\nu} = \sum_{\mu=1}^{s_3} \|M_{\nu i\mu}\|^2 \|N_{\nu i\mu}\|^2, \qquad \Theta_{4\nu} = \sum_{\mu=1}^{s_4} \|H_{\nu i\mu}\|^2 \|G_{\nu i\mu}\|^2.$$

Then the iterative solution groups $X^{(t)} = (X_1^{(t)}, X_2^{(t)}, \ldots, X_p^{(t)})$, $t = 1, 2, 3, \ldots$, *computed by Algorithm 1 (with* $\kappa = 0$*), converge for any initial reflexive matrix group*

$$X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \ldots, X_p^{(1)}).$$

Proof. Without loss of generality, we may assume that $\alpha_{ij} = 1$ for $i \in I[1,p]$ and $j = 1,2,3,4$. For $i = 1,2,\ldots,p$, we set

$$\widetilde{U}_i^{(t)} = -\frac{U_i^{(t)} - X_i^*}{\beta_i}, \widetilde{Y}_i^{(t)} = -\frac{Y_i^{(t)} - X_i^*}{\beta_i}, \widetilde{Z}_i^{(t)} = -\frac{Z_i^{(t)} - X_i^*}{\beta_i}, \widetilde{W}_i^{(t)} = -\frac{W_i^{(t)} - X_i^*}{\beta_i}.$$

For the matrix groups $U = (U_1, U_2, \ldots, U_p)$, $Y = (Y_1, Y_2, \ldots, Y_p)$, $Z = (Z_1, Z_2, \ldots, Z_p)$ and $W = (W_1, W_2, \ldots, W_p)$, we define

$$T_\nu(U, Y, Z, W) = \mathcal{T}_\nu(L), \quad \nu = 1, 2, \ldots, N,$$

where the matrix groups $L = (L_1, L_2, \ldots, L_p)$ is defined such that $L_i = (U_i + Y_i + Z_i + W_i)$ for $i \in I[1,p]$.

It is obvious that $\widetilde{L}^{(t)} = X^* - X^{(t)}$. Therefore,

$$T_\nu(\widetilde{U}^{(t)}, \widetilde{Y}^{(t)}, \widetilde{Z}^{(t)}, \widetilde{W}^{(t)}) = \mathcal{T}_\nu(\widetilde{L}^{(t)}) = \mathcal{T}_\nu(X^*) - \mathcal{T}_\nu(X^{(t)}) = R_\nu^{(t)}.$$

Suppose that the matrices $R_i, Q_i \in \mathbb{SOC}^{p_i \times p_i}$ for $i \in I[1,p]$ are given. Assume that t steps of Algorithm 1, with $\kappa = 0$, has been performed. Evidently, the t-th approximate solution group $X^{(t)} = (X_1^{(t)}, X_2^{(t)}, \ldots, X_p^{(t)})$ is a reflexive matrix group, i.e. $X_i^{(t)} \in \mathbb{C}^{p_i \times p_i}(R_i, Q_i)$ for $i \in I[1,p]$. Hence, $\widetilde{L}^{(t)} = (\widetilde{L}_1^{(t)}, \widetilde{L}_2^{(t)}, \ldots, \widetilde{L}_p^{(t)})$ is a group of reflexive matrices, i.e., $\widetilde{L}_i^{(t)} \in \mathbb{C}_r^{p_i \times p_i}(R_i, Q_i)$ for $i \in I[1,p]$. In addition, for two given arbitrary matrices A and B, it is not difficult to see that $\mathrm{Re}[\mathrm{tr}(AB)] = \mathrm{Re}[\mathrm{tr}(BA)] = \mathrm{Re}[\mathrm{tr}(A^H B^H)]$. Therefore, we get

$$\left\|\widetilde{U}^{(t)}\right\|^2 = \mathrm{Re}[tr[\textstyle\sum_{i=1}^p (\widetilde{U}_i^{(t)})^H \widetilde{U}_i^{(t)}]] = \frac{1}{16 s_1^2} \left\|\widetilde{L}^{(t-1)}\right\|^2$$

$$-\frac{\omega}{4Ns_1^2}\left\{\textstyle\sum_{i=1}^p \mathrm{Re}\left[tr\left((\widetilde{L}_i^{(t-1)})^H \left(\sum_{\nu=1}^N \sum_{\mu=1}^{s_1} A_{\nu i\mu}^H T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) B_{\nu i\mu}^H\right.\right.\right.\right.$$

$$\left.\left.\left.\left. + R_i A_{\nu i\mu}^H T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) B_{\nu i\mu}^H Q_i\right)\right)\right]\right\}$$

$$+\frac{\omega^2}{16N^2 s_1^2}\left\{\textstyle\sum_{i=1}^p \left\|\sum_{\nu=1}^N \sum_{\mu=1}^{s_1} A_{\nu i\mu}^H T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) B_{\nu i\mu}^H\right.\right.$$

$$\left.\left. + R_i A_{\nu i\mu}^H T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) B_{\nu i\mu}^H Q_i\right\|^2\right\}$$

$$\leq \frac{1}{16 s_1^2}\left\|\widetilde{L}^{(t-1)}\right\|^2 - \frac{\omega}{4Ns_1^2}\textstyle\sum_{\nu=1}^N \sum_{\mu=1}^{s_1}$$

$$\left\{\mathrm{Re}\left[tr\left(\left(\textstyle\sum_{i=1}^p A_{\nu i\mu}\widetilde{L}_i^{(t-1)} B_{\nu i\mu}\right)\left(T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)})\right)^H\right)\right]\right.$$
$$\left. + \mathrm{Re}\left[tr\left(\left(\textstyle\sum_{i=1}^p A_{\nu i\mu} R_i \widetilde{L}_i^{(t-1)} Q_i B_{\nu i\mu}\right)\left(T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)})\right)^H\right)\right]\right\}$$

$$+\frac{\omega^2}{2Ns_1^2}\left\{\textstyle\sum_{i=1}^p \sum_{\nu=1}^N \sum_{\mu=1}^{s_1}\right\|A_{\nu i\mu}^H T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) B_{\nu i\mu}^H$$

$$+R_iA_{\nu i\mu}^{H}T_\nu(\widetilde{U}^{(t-1)},\widetilde{Y}^{(t-1)},\widetilde{Z}^{(t-1)}\ ,\widetilde{W}^{(t-1)})B_{\nu i\mu}^{H}Q_i\Big\|^2\Big\} \leq \tfrac{1}{16s_1^2}\left\|\widetilde{L}^{(t-1)}\right\|^2$$
$$-\tfrac{\omega}{2Ns_1^2}\left\{\sum_{\nu=1}^{N}\sum_{\mu=1}^{s_1}\mathrm{Re}\left[tr\left(\left(\ \sum_{i=1}^{p}A_{\nu i\mu}\widetilde{L}_i^{(t-1)}B_{\nu i\mu}\ \right)\left(T_\nu(\widetilde{U}^{(t-1)},\widetilde{Y}^{(t-1)},\widetilde{Z}^{(t-1)},\widetilde{W}^{(t-1)})\right)^H\right)\right]\right\}$$
$$+\tfrac{\omega^2}{4Ns_1^2}\left\{\sum_{i=1}^{p}\sum_{\nu=1}^{N}\sum_{\mu=1}^{s_1}\left\|A_{\nu i\mu}\right\|^2\left\|B_{\nu i\mu}\right\|^2\left\|T_\nu(\widetilde{U}^{(t-1)},\widetilde{Y}^{(t-1)},\widetilde{Z}^{(t-1)},\widetilde{W}^{(t-1)})\right\|^2\right\}.$$

Similarly, we can see that

$$\left\|\widetilde{Y}^{(t)}\right\|^2 \leq \tfrac{1}{16s_2^2}\left\|\widetilde{L}^{(t-1)}\right\|^2$$
$$-\tfrac{\omega}{2Ns_2^2}\left\{\sum_{\nu=1}^{N}\sum_{\mu=1}^{s_2}\mathrm{Re}\left[tr\left(\left(\ \sum_{i=1}^{p}C_{\nu i\mu}\widetilde{L}_i^{(t-1)}D_{\nu i\mu}\ \right)\left(T_\nu(\widetilde{U}^{(t-1)},\widetilde{Y}^{(t-1)},\widetilde{Z}^{(t-1)},\widetilde{W}^{(t-1)})\right)^H\right)\right]\right\}$$
$$+\tfrac{\omega^2}{4Ns_2^2}\left\{\sum_{i=1}^{p}\sum_{\nu=1}^{N}\sum_{\mu=1}^{s_2}\left\|C_{\nu i\mu}\right\|^2\left\|D_{\nu i\mu}\right\|^2\left\|T_\nu(\widetilde{U}^{(t-1)},\widetilde{Y}^{(t-1)},\widetilde{Z}^{(t-1)},\widetilde{W}^{(t-1)})\right\|^2\right\},$$

$$\left\|\widetilde{Z}^{(t)}\right\|^2 \leq \tfrac{1}{16s_3^2}\left\|\widetilde{L}^{(t-1)}\right\|^2$$
$$-\tfrac{\omega}{2Ns_3^2}\left\{\sum_{\nu=1}^{N}\sum_{\mu=1}^{s_3}\mathrm{Re}\left[tr\left(\left(\ \sum_{i=1}^{p}M_{\nu i\mu}\widetilde{L}_i^{(t-1)}N_{\nu i\mu}\ \right)\left(T_\nu(\widetilde{U}^{(t-1)},\widetilde{Y}^{(t-1)},\widetilde{Z}^{(t-1)},\widetilde{W}^{(t-1)})\right)^H\right)\right]\right\}$$
$$+\tfrac{\omega^2}{4Ns_3^2}\left\{\sum_{i=1}^{p}\sum_{\nu=1}^{N}\sum_{\mu=1}^{s_3}\left\|M_{\nu i\mu}\right\|^2\left\|N_{\nu i\mu}\right\|^2\left\|T_\nu(\widetilde{U}^{(t-1)},\widetilde{Y}^{(t-1)},\widetilde{Z}^{(t-1)},\widetilde{W}^{(t-1)})\right\|^2\right\},$$

and,

$$\left\|\widetilde{W}^{(t)}\right\|^2 \leq \tfrac{1}{16s_4^2}\left\|\widetilde{L}^{(t-1)}\right\|^2$$
$$-\tfrac{\omega}{2Ns_4^2}\left\{\sum_{\nu=1}^{N}\sum_{\mu=1}^{s_4}\mathrm{Re}\left[tr\left(\left(\ \sum_{i=1}^{p}H_{\nu i\mu}\widetilde{L}_i^{(t-1)}G_{\nu i\mu}\ \right)\left(T_\nu(\widetilde{U}^{(t-1)},\widetilde{Y}^{(t-1)},\widetilde{Z}^{(t-1)},\widetilde{W}^{(t-1)})\right)^H\right)\right]\right\}$$
$$+\tfrac{\omega^2}{4Ns_4^2}\left\{\sum_{i=1}^{p}\sum_{\nu=1}^{N}\sum_{\mu=1}^{s_4}\left\|H_{\nu i\mu}\right\|^2\left\|G_{\nu i\mu}\right\|^2\left\|T_\nu(\widetilde{U}^{(t-1)},\widetilde{Y}^{(t-1)},\widetilde{Z}^{(t-1)},\widetilde{W}^{(t-1)})\right\|^2\right\}.$$

Assume that

$$\Lambda^{(t)} = 4N\left(s_1^2\left\|\widetilde{U}^{(t)}\right\|^2 + s_2^2\left\|\widetilde{Y}^{(t)}\right\|^2 + s_3^2\left\|\widetilde{Z}^{(t)}\right\|^2 + s_4^2\left\|\widetilde{W}^{(t)}\right\|^2\right).$$

By some straightforward computations, we get

$$\begin{aligned}
&0 \leq \Lambda^{(t)} \leq \Lambda^{(t-1)} \\
&-2\omega \Bigg\{ \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_1} \mathrm{Re}\left[tr\left(\left(\sum_{i=1}^{p} A_{\nu i\mu} \widetilde{L}_i^{(t-1)} B_{\nu i\mu} \right) \left(T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) \right)^H \right) \right] \\
&+ \sum_{\mu=1}^{s_2} \mathrm{Re}\left[tr\left(\left(\sum_{i=1}^{p} C_{\nu i\mu} \widetilde{L}_i^{(t-1)} D_{\nu i\mu} \right) \left(T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) \right)^H \right) \right] \\
&+ \sum_{\mu=1}^{s_3} \mathrm{Re}\left[tr\left(\left(\sum_{i=1}^{p} M_{\nu i\mu} \widetilde{L}_i^{(t-1)} N_{\nu i\mu} \right) \left(T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) \right)^H \right) \right] \\
&+ \sum_{\mu=1}^{s_4} \mathrm{Re}\left[tr\left(\left(\sum_{i=1}^{p} H_{\nu i\mu} \widetilde{L}_i^{(t-1)} G_{\nu i\mu} \right) \left(T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) \right)^H \right) \right] \Bigg\} \\
&+\omega^2 \Bigg\{ \sum_{i=1}^{p} \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_1} \left\| A_{\nu i\mu} \right\|^2 \left\| B_{\nu i\mu} \right\|^2 \left\| T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) \right\|^2 \\
&+ \sum_{i=1}^{p} \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_2} \left\| C_{\nu i\mu} \right\|^2 \left\| D_{\nu i\mu} \right\|^2 \left\| T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) \right\|^2 \\
&\quad + \sum_{i=1}^{p} \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_3} \left\| M_{\nu i\mu} \right\|^2 \left\| N_{\nu i\mu} \right\|^2 \left\| T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) \right\|^2 \\
&\quad + \sum_{i=1}^{p} \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_4} \left\| H_{\nu i\mu} \right\|^2 \left\| G_{\nu i\mu} \right\|^2 \left\| T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) \right\|^2 \Bigg\} \\
&\quad \leq \Lambda^{(t-1)} - 2\omega \sum_{\nu=1}^{N} \left\| T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) \right\|^2 \\
&\quad +\omega^2 \Bigg\{ \sum_{i=1}^{p} \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_1} \left\| A_{\nu i\mu} \right\|^2 \left\| B_{\nu i\mu} \right\|^2 + \sum_{i=1}^{q} \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_2} \left\| C_{\nu i\mu} \right\|^2 \left\| D_{\nu i\mu} \right\|^2 \\
&\quad + \sum_{i=1}^{p} \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_3} \left\| M_{\nu i\mu} \right\|^2 \left\| N_{\nu i\mu} \right\|^2 + \sum_{i=1}^{p} \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_4} \left\| H_{\nu i\mu} \right\|^2 \left\| G_{\nu i\mu} \right\|^2 \Bigg\} \\
&\quad \times \sum_{\nu=1}^{N} \left\| T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) \right\|^2.
\end{aligned}$$

As ω satisfies in (3.4), by some straightforward computations, we have:

$$\begin{aligned}
&\Lambda^{(t)} \leq \Lambda^{(t-1)} - \omega \Bigg(2 - \omega \Bigg\{ \sum_{i=1}^{p} \sum_{\nu=1}^{N} \sum_{\mu=1}^{s_1} \left\| A_{\nu i\mu} \right\|^2 \left\| B_{\nu i\mu} \right\|^2 + \sum_{\mu=1}^{s_2} \left\| C_{\nu i\mu} \right\|^2 \left\| D_{\nu i\mu} \right\|^2 \\
&+ \sum_{\mu=1}^{s_3} \left\| M_{\nu i\mu} \right\|^2 \left\| N_{\nu i\mu} \right\|^2 + \sum_{\mu=1}^{s_4} \left\| H_{\nu i\mu} \right\|^2 \left\| G_{\nu i\mu} \right\|^2 \Bigg\} \Bigg) \times \sum_{\nu=1}^{N} \left\| T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) \right\|^2 \\
&= \Lambda^{(t-1)} - \omega\,(2 - \omega\ \Theta) \times \sum_{\nu=1}^{N} \left\| T_\nu(\widetilde{U}^{(t-1)}, \widetilde{Y}^{(t-1)}, \widetilde{Z}^{(t-1)}, \widetilde{W}^{(t-1)}) \right\|^2 \\
&\leq \Lambda^{(0)} - \omega\,(2 - \omega\ \Theta) \times \sum_{\rho=0}^{t-1} \sum_{\nu=1}^{N} \left\| T_\nu(\widetilde{U}^{(\rho-1)}, \widetilde{Y}^{(\rho-1)}, \widetilde{Z}^{(\rho-1)}, \widetilde{W}^{(\rho-1)}) \right\|^2.
\end{aligned}$$

Therefore,

$$0 \leq \omega\,(2 - \omega\ \Theta) \times \sum_{\rho=0}^{t-1} \sum_{\nu=1}^{N} \left\| T_\nu(\widetilde{U}^{(\rho-1)}, \widetilde{Y}^{(\rho-1)}, \widetilde{Z}^{(\rho-1)}, \widetilde{W}^{(\rho-1)}) \right\|^2 \leq \Lambda^{(0)},$$

which shows that

$$0 \leq \omega\,(2-\omega\ \Theta) \times \sum_{\rho=0}^{\infty} \sum_{\nu=1}^{N} \left\| T_\nu(\widetilde{U}^{(\rho-1)}, \widetilde{Y}^{(\rho-1)}, \widetilde{Z}^{(\rho-1)}, \widetilde{W}^{(\rho-1)}) \right\|^2 \leq \Lambda^{(0)} < \infty.$$

From the convergence theorem of series, we conclude that

$$\lim_{t\to\infty} \sum_{\nu=1}^{N} \left\| T_\nu(\widetilde{U}^{(t)}, \widetilde{Y}^{(t)}, \widetilde{Z}^{(t)}, \widetilde{W}^{(t)}) \right\|^2 = 0.$$

Hence,

$$\lim_{t\to\infty} T_\nu(\widetilde{U}^{(t)}, \widetilde{Y}^{(t)}, \widetilde{Z}^{(t)}, \widetilde{W}^{(t)}) = 0.$$

Or equivalency,

$$\lim_{t\to\infty} R_\nu^{(t)} = 0, \qquad \nu = 1, 2, \ldots, N.$$

From Theorem 3.3, we can conclude the result immediately. □

With a same strategy employed in the proof of Theorem 3.5, we may establish the following theorem.

Theorem 3.6. *Suppose that the linear coupled matrix equations* (1.4) *has unique anti-reflexive solution group* $X^* = (X_1^*, X_2^*, \ldots, X_p^*)$. *If the parameter* ω *satisfies the inequality*

$$0 < \omega < \frac{2}{\Theta}, \tag{3.4}$$

such that $\Theta = \sum_{\nu=1}^{N} \sum_{i=1}^{p} \Theta_{i\nu}$, *where,*

$$\Theta_{1\nu} = \sum_{\mu=1}^{s_1} \|A_{\nu i\mu}\|^2 \|B_{\nu i\mu}\|^2, \qquad \Theta_{2\nu} = \sum_{\mu=1}^{s_2} \|C_{\nu i\mu}\|^2 \|D_{\nu i\mu}\|^2$$
$$\Theta_{3\nu} = \sum_{\mu=1}^{s_3} \|M_{\nu i\mu}\|^2 \|N_{\nu i\mu}\|^2, \qquad \Theta_{4\nu} = \sum_{\mu=1}^{s_4} \|H_{\nu i\mu}\|^2 \|G_{\nu i\mu}\|^2.$$

Then the iterative solution groups $X^{(t)} = (X_1^{(t)}, X_2^{(t)}, \ldots, X_p^{(t)})$, $t = 1, 2, 3, \ldots$, *computed by Algorithm 1 (with* $\kappa = 1$*), converge for any initial anti-reflexive matrix group*

$$X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \ldots, X_p^{(1)}).$$

4. Numerical experiments

In this section, we present two numerical examples to show the effectiveness of the proposed algorithm. All the numerical experiments presented in this section were computed in double precision with some Matlab codes on a Pentium 4 PC, with a 3.06 GHz CPU and 1.00GB of RAM.

Example 4.1. We consider the following matrix equation

$$AXB + CX^TD + M\bar{X}N + HX^HG = F, \tag{4.1}$$

where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -3 & 1 \\ 1 & 2 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} i & 1 & 1 \\ -1 & 1 & 1 \\ i & 2 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & -i & 1 \\ i & 1 & 1 \\ i & i & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & i & 1 \\ 1+i & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 4 & 1 \\ -2 & 0 & 2 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 & -1 \\ 0 & i & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

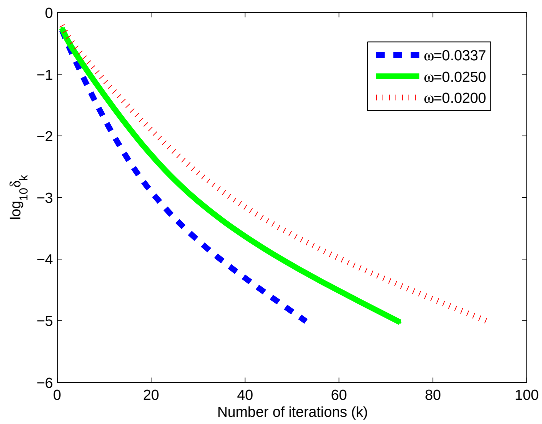

FIGURE 1. $\log_{10} \delta_k$ versus k for the (R, Q)-reflexive solution of (4.1) .

.

and

$$F = \begin{pmatrix} -6+14i & 1-5i & 37-3i \\ 7+7i & -3-4i & 18-1i \\ -4+11i & -5+13i & 10-1i \end{pmatrix}.$$

The exact solution of (4.1) is

$$X^* = \begin{pmatrix} 2+2i & 2-2i & 1 \\ 1 & -1 & 2i \\ 2-2i & 2+2i & -1 \end{pmatrix},$$

which is (R, Q)-reflexive, where

$$R = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.2}$$

Here, we mention that $R, Q \in \mathbb{SOC}^{3\times 3}$.

We apply Algorithm 1 with $\kappa = 0$ to solve Problem 1 corresponding to system (4.1). The initial guess was taken to be the zero matrix $X_1^{(0)} = 0$ and the stopping criterion

$$\delta_k = \frac{\|R_k\|_F}{\|R_0\|_F} < 10^{-5},$$

was used where R_k is the residual of (4.1) at kth iteration. For $\omega = 0.0337$, 0250, 0.0200 the method converges in 53, 73 and 92 iterations, respectively. For $\omega = 0.0337$, the computed solution is

$$X^{(53)} = \begin{pmatrix} 2.00000+2.00002i & 2.00002-1.99996i & 1.00000-0.00007i \\ 0.99999-0.00001i & -0.99999+0.00001i & 0.00003+2.00000i \\ 2.00002-1.99996i & 2.00000+2.00002i & -1.00000+0.00007i \end{pmatrix},$$

which is in good agreement with the exact solution. In FIGURE 1 the convergence history of the method is displayed.

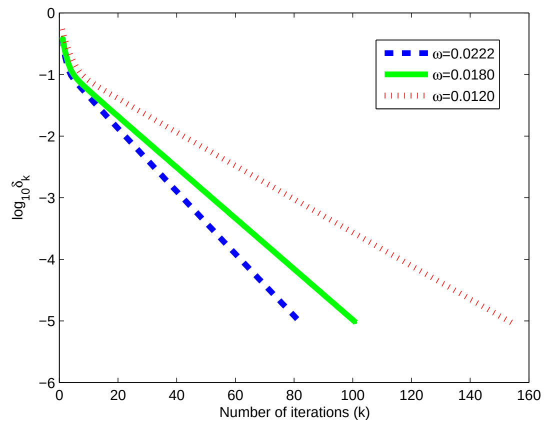

FIGURE 2. $\log_{10}\delta_k$ versus k for the (R,Q)-anti-reflexive solution of (4.1) .

.

We now consider (4.1) with the right-hand side

$$F=\begin{pmatrix} 16i & 3+19i & -5-31i \\ 3+11i & 1+12i & -3i \\ 6+15i & 13+9i & 4-11i \end{pmatrix}.$$

The exact solution of the system is

$$X^*=\begin{pmatrix} 0 & 0 & 1+2i \\ 1-2i & 1-2i & 0 \\ 0 & 0 & 1+2i \end{pmatrix}.$$

which is in $\mathbb{C}_a^{3\times 3}(R,Q)$ where R and Q are defined by (4.2). All of the other assumptions are as before. Performing Algorithm 1 to the matrix equation with $\kappa=1$, the method converges in 82, 101 and 154 iterations for $\omega=0.0222$, 0.0180, 0.0120, respectively. For $\omega=0.0222$ the computed solution is

$$X^{(82)}=\begin{pmatrix} -0.00001+0.00001i & 0.00002+0.00003i & 0.99999+2.00002i \\ 0.99999-1.99995i & 0.99999-1.99995i & 0.00000+0.00000i \\ -0.00002-0.00003i & 0.00001-0.00001i & 0.99999+2.00002i \end{pmatrix}.$$

We observe that the method has provided a good approximate solution for the system. The convergence history is displayed in FIGURE 2.

Example 4.2. In this example, we consider the system of matrix equations

$$\text{(4.3)}\qquad \begin{cases} X_1+A_1X_1B_1+M_1\bar{X}_1N_1+C_2X_2^TD_2+H_2X_2^HG_2= & F_1,\\ X_1+\tilde{A}_1X_1\tilde{B}_1+\tilde{M}_1\bar{X}_1\tilde{N}_1+\tilde{C}_2X_2^T\tilde{D}_2+\tilde{H}_2X_2^H\tilde{G}_2= & F_2, \end{cases}$$

where

$$A_1 = \begin{pmatrix} 1 & 0 & 3 \\ -1 & -3 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -3 & 1 \\ 1 & 2 & -3 \end{pmatrix}, \quad M_1 = \begin{pmatrix} i & 1 & 1 \\ -1 & 1 & 1 \\ i & 2 & 1 \end{pmatrix}$$

$$N_1 = \begin{pmatrix} 1 & -i & 1 \\ i & 1 & 1 \\ 2 & i & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & i & 1 \\ 2i & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 4 & 1 \\ -2 & 0 & -2 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & i & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_1 = \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$\tilde{B}_1 = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -3 & 1 \\ 1 & 2 & -3 \end{pmatrix}, \quad \tilde{M}_1 = \begin{pmatrix} i & 0 & 1 \\ -1 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}, \quad \tilde{N}_1 = \begin{pmatrix} 1 & -i & 1 \\ 1 & 1 & 1 \\ i & i & 1 \end{pmatrix}$$

$$\tilde{C}_2 = \begin{pmatrix} 1 & -i & 1 \\ i & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \quad \tilde{D}_2 = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -3 \end{pmatrix}, \quad \tilde{H}_2 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 4 & 1 \\ -2 & 0 & 2 \end{pmatrix},$$

$$\tilde{G}_2 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -i & 1 \\ -1 & 2 & 1 \end{pmatrix},$$

and

$$F_1 = \begin{pmatrix} 83+67i & 88+66i & 132+61i \\ 58+183i & 41-3i & -36-58i \\ -13+76i & 54-90i & 120-126i \end{pmatrix},$$

$$F_2 = \begin{pmatrix} 9+29i & 74-67i & -12+95i \\ 68+190i & 99-36i & -118-215i \\ 64+85i & -47-55i & -32-274i \end{pmatrix}.$$

The exact solution of Eq. (4.3) is the matrix group (X_1^*, X_2^*) where

$$X_1^* = \begin{pmatrix} 2+2i & 2-2i & 1 \\ 1 & -1 & 2i \\ 2-2i & 2+2i & -1 \end{pmatrix}, \quad X_2^* = \begin{pmatrix} 16+16i & 20-28i & 12+4i \\ 29-4i & -7+5i & -4+17i \\ 10-19i & 9+11i & -1-2i \end{pmatrix}.$$

Here, $X_1^* \in \mathbb{C}_r^{3\times3}(R_1, Q_1)$ where $R_1 = R$ and $Q_1 = Q$ are as in previous example and $X_2^* \in \mathbb{C}_r^{3\times3}(R_2, Q_2)$, where

$$R_2 = \frac{1}{5}\begin{pmatrix} 1 & 4i & -2+2i \\ -4i & 1 & -2-2i \\ -2-2i & -2+2i & 3 \end{pmatrix},$$

$$Q_2 = \frac{1}{2}\begin{pmatrix} 1 & -1-1i & -1i \\ -1+1i & 0 & -1-1i \\ 1i & -1+1i & 1 \end{pmatrix}.$$

We apply Algorithm 1 with $\kappa = 0$ for solving system (4.3). The initial guess was taken to be the zero matrix group $(X_1^{(0)}, X_2^{(0)}) = (0, 0)$ and the stopping criterion

$$\delta_k = \max\{\frac{\|R_i^{(k)}\|_F}{\|R_i^{(0)}\|_F} : i = 1, 2\} < 10^{-5},$$

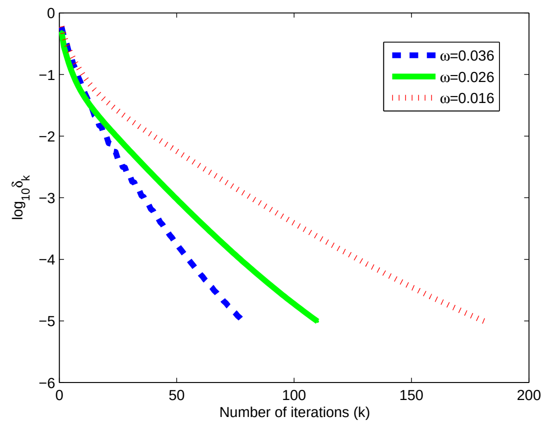

FIGURE 3. $\log_{10}\delta_k$ versus k for the (R_i, Q_i)-reflexive of (4.3) .

was used, where $R_k^{(i)}$ is the residual of the ith equation of (4.3) at iteration k. For $\omega = 0.036$, $w = 0.026$, 0.016 the method, respectively, converges in 79, 110 and 181 iterations. For $\omega = 0.036$ the computed solution is $(X_1^{(79)}, X_2^{(79)})$, where

$$X_1^{(79)} = \begin{pmatrix} 1.99988+2.00028i & 1.99951-1.99997i & 1.00011-0.00021i \\ 0.99986+0.00016i & -0.99986-0.00016i & 0.00009+1.99983i \\ 1.99951-1.99997i & 1.99988+2.00028i & -1.00011+0.00021i \end{pmatrix},$$
$$X_2^{(79)} = \begin{pmatrix} 16.00000+16.00030i & 19.99987-27.99978i & 12.00068+3.99996i \\ 28.99993-3.99999i & -6.99963+5.00038i & -4.00003+16.99934i \\ 10.00041-19.00036i & 8.99963+10.99996i & -1.00005-1.99997i \end{pmatrix}.$$

As seen the approximate solution group $(X_1^{(79)}, X_2^{(79)})$ is in good agreement with the exact solution group. The corresponding results are also depicted in FIGURE 3.

We now change the right-hand side of (4.3) to

$$F_1 = \frac{1}{10}\begin{pmatrix} 148+24i & 188+144i & -22-196i \\ -22-124i & 142-44i & 148+428i \\ 77+104i & 159+140i & 59-100i \end{pmatrix},$$
$$F_2 = \frac{1}{10}\begin{pmatrix} 116+74i & 142+232i & -58-150i \\ 27-219i & 374+406i & -249+423i \\ -12+68i & 208+130i & -16+78i \end{pmatrix}.$$

In this case, the exact solution of the system is (X_1^*, X_2^*), where

$$X_1^* = \begin{pmatrix} 0 & 0 & 1+2i \\ 1-2i & 1-2i & 0 \\ 0 & 0 & 1+2i \end{pmatrix}, \quad X_2^* = \frac{1}{10}\begin{pmatrix} 18-20i & 6+14i & 6i \\ 5+8i & -19-19i & 4-15i \\ 9-26i & 11-25i & 22-7i \end{pmatrix}.$$

We have $X_i^* \in \mathbb{C}_a^{3\times 3}(R_i, Q_i)$, for $i = 1, 2$. All of the assumptions are as before. Algorithm 1 with $k = 1$ converges in 180, 223 and 336 iteration, respectively, for $\omega = 0.0222$, 0.0180,

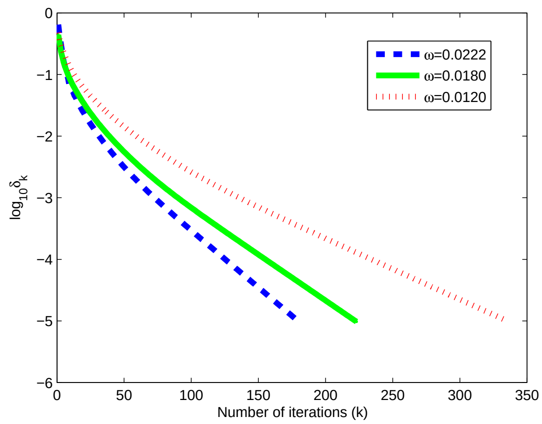

FIGURE 4. $\log_{10}\delta_k$ versus k for the (R_i, Q_i)-anti-reflexive of (4.3) .

0.0120. For $w = 0.0222$ the computed solution by Algorithm 1 is $(X_1^{(180)}, X_2^{(180)})$ where

$$X_1^{(180)} = \begin{pmatrix} 0.00003 + 0.00009i & 0.00003 + 0.00010i & 0.99998 + 1.99999i \\ 0.99997 - 1.99999i & 0.99997 - 1.99999i & 0.00000 + 0.00000i \\ -0.00003 - 0.00010i & -0.00003 - 0.00009i & 0.99998 + 1.99999i \end{pmatrix},$$

$$X_2^{(180)} = \begin{pmatrix} 1.79996 - 2.00002i & 0.60003 + 1.40002i & -0.00001 + 0.60000i \\ 0.50000 + 0.79996i & -1.90004 - 1.89999i & 0.40000 - 1.50003i \\ 0.89999 - 2.60004i & 1.10001 - 2.49999i & 2.20000 - 0.70001i \end{pmatrix}.$$

The convergence history of the method is shown in FIGURE 4.

5. CONCLUSION

This paper has been devoted for finding the reflexive (anti-reflexive) solution group of a general class of complex coupled linear matrix equations. To this end, using hierarchical identification principle, an iterative algorithm has been constructed. We have established that the proposed algorithm converges to the reflexive (anti-reflexive) solution group of the mentioned coupled linear matrix equations for any initial reflexive (anti-reflexive) solution group. In order to illustrate the feasibly and effectively of the presented algorithm, some numerical experiments have been given.

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