NUMERICAL LINEAR ALGEBRA WITH APPLICATIONS
Numer. Linear Algebra Appl. 2000; **00**:1–6 *Prepared using* nlaauth.cls *[Version: 2002/09/18 v1.02]*

Letter to the Editor regarding "Modified Hermitian and skew-Hermitian splitting methods for non-Hermitian positive-definite linear systems" [Numer. Linear Algebra Appl. 14 (2007) 217-235]

Davod Khojasteh Salkuyeh*and Shiva Behnejad

Department of Mathematics, University of Mohaghegh Ardabili, P.O. Box 179, Ardabil, Iran

SUMMARY

In this note, some errors in the article [L. Li, T. Z. Huang and X. P. Liu, Modified Hermitian and skew-Hermitian splitting methods for non-Hermitian positive-definite linear systems, Numer. Linear Algebra Appl. 14 (2007) 217-235] are pointed out and some correct results are presented. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: non-Hermitian positive-definite matrix; skew-Hermitian matrix; splitting; lopsided

1. INTRODUCTION

Consider the system of linear equations

$$Ax = b \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$ is a large sparse non-Hermitian positive-definite matrix and $x, b \in \mathbb{C}^n$. Let

$$A = H + S$$

where

$$H = \frac{1}{2}(A + A^*) \qquad \text{and} \qquad S = \frac{1}{2}(A - A^*)$$

are the Hermitian and skew-Hermitian parts of A, respectively. Obviously H is a Hermitian positive definite matrix. To solve (1), in [1], Bai et al. presented the Hermitian/skew-Hermitian splitting iteration, or briefly, the HSS iteration method as follows:

*Correspondence to: D. K. Salkuyeh, Department of Mathematics, University of Mohaghegh Ardabili, P.O. Box 179, Ardabil, Iran.
†khojaste@uma.ac.ir

Received
Revised

The HSS iteration method. `Given an initial guess` $x^{(0)}$. `For` $k = 0, 1, 2, \ldots$ `until` $x^{(k)}$ `converges, compute`

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases}$$

`where` α `is a given positive constant.`

In [1], It was shown that the HSS iteration is unconditionally convergent, i.e., the sequence $\{x^{(k)}\}$ convergence to the solution $x_* = A^{-1}b$ as $k \to \infty$ for all $\alpha > 0$ and for any choice of x_0. In [2], Li et al. proposed the LHSS iteration (for Lopsided Hermitian/skew-Hermitian splitting iteration). This algorithm is described as follows.

The LHSS iteration method. `Given an initial guess` $x^{(0)}$. `For` $k = 0, 1, 2, \ldots$ `until` $x^{(k)}$ `converges, compute`

$$\begin{cases} Hx^{(k+\frac{1}{2})} = -Sx^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases}$$

`where` α `is a given non-zero constant.`

In [2], the authors investigated the convergence properties of the LHSS iteration. There are some errors in both theoretical and numerical results presented in [2] and in this note we correct them. For convenience, we use the same notations as in [2].

2. ERRORS AND CORRECTIONS

The second part of *Corollary* 2.3 in [2] is not correct and we correct it as follows.

Corollary 2.3
Let A, H and S be defined as those in Theorem 2.2, and $\lambda_{\max}$, $\lambda_{\min}$ be the maximum and minimum eigenvalues of the matrix H, $\sigma_{\max}$ be the maximum singular value of the matrix S. Then the optimal parameter α is

$$\alpha^* = \frac{2\lambda_{\min}\lambda_{\max}}{\lambda_{\min} + \lambda_{\max}},$$

and the bound for $\rho(M(\alpha))$ is

$$\delta(\alpha^*) = \frac{(\lambda_{\max} - \lambda_{\min})\sigma_{\max}}{\sqrt{4\lambda_{\min}^2\lambda_{\max}^2 + \sigma_{\max}^2(\lambda_{\max} + \lambda_{\min})^2}} < 1.$$

Proof

For the second part of this theorem, we have

$$\begin{aligned}\delta(\alpha^*) &= \frac{\sigma_{\max}}{\sqrt{\alpha^{*2}+\sigma_{\max}^2}}\frac{\lambda_{\max}-\alpha^*}{\lambda_{\max}}\\ &= \frac{\sigma_{\max}}{\sqrt{(\frac{2\lambda_{\min}\lambda_{\max}}{\lambda_{\min}+\lambda_{\max}})^2+\sigma_{\max}^2}}\frac{\lambda_{\max}-\frac{2\lambda_{\min}\lambda_{\max}}{\lambda_{\min}+\lambda_{\max}}}{\lambda_{\max}}\\ &= \frac{\sigma_{\max}(\lambda_{\max}-\lambda_{\min})}{\sqrt{4\lambda_{\min}^2\lambda_{\max}^2+\sigma_{\max}^2(\lambda_{\max}+\lambda_{\min})^2}}. \qquad \square\end{aligned}$$

Theorem 2.5 in [2] is not correct and should be rectified as follows.

Theorem 2.5
Let $\lambda_{\min}$, $\lambda_{\max}$, $\sigma_{\max}$, α^* and $\delta(\alpha^*)$ be defined as those in Corollary 2.3 and $\bar{\alpha}$ and $\gamma(\bar{\alpha})$ be those in Lemma 2.4, then, if

$$\sigma_{\max}^2 \leq \frac{\lambda_{\min}^2\lambda_{\max}^2}{(\lambda_{\max}+\lambda_{\min})\sqrt{\lambda_{\max}\lambda_{\min}}+\lambda_{\max}\lambda_{\min}} \tag{2}$$

then the following inequality holds:

$$\delta(\alpha^*) \leq \gamma(\bar{\alpha}).$$

Proof
Suppose that $\delta(\alpha^*) \leq \gamma(\bar{\alpha})$. Then

$$\frac{(\lambda_{\max}-\lambda_{\min})\sigma_{\max}}{\sqrt{4\lambda_{\min}^2\lambda_{\max}^2+\sigma_{\max}^2(\lambda_{\max}+\lambda_{\min})^2}} \leq \frac{\sqrt{\lambda_{\max}}-\sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}}+\sqrt{\lambda_{\min}}}.$$

This implies

$$\frac{\sigma_{\max}}{\sqrt{4\lambda_{\min}^2\lambda_{\max}^2+\sigma_{\max}^2(\lambda_{\max}+\lambda_{\min})^2}} \leq \frac{1}{(\sqrt{\lambda_{\max}}+\sqrt{\lambda_{\min}})^2}$$

and therefore

$$(\sqrt{\lambda_{\max}}+\sqrt{\lambda_{\min}})^2\sigma_{\max} \leq \sqrt{4\lambda_{\min}^2\lambda_{\max}^2+\sigma_{\max}^2(\lambda_{\max}+\lambda_{\min})^2}.$$

Then, squaring both sides and combining the coefficients of $\sigma_{\max}$, we obtain

$$\begin{aligned}&(\sqrt{\lambda_{\max}}+\sqrt{\lambda_{\min}})^4{\sigma_{\max}}^2 \leq 4\lambda_{\min}^2\lambda_{\max}^2+\sigma_{\max}^2(\lambda_{\max}+\lambda_{\min})^2\\ \Longrightarrow\quad &((\sqrt{\lambda_{\max}}+\sqrt{\lambda_{\min}})^4-(\lambda_{\max}+\lambda_{\min})^2){\sigma_{\max}}^2 \leq 4\lambda_{\min}^2\lambda_{\max}^2\\ \Longrightarrow\quad &((\lambda_{\max}+\lambda_{\min}+2\sqrt{\lambda_{\max}\lambda_{\min}})^2-(\lambda_{\max}+\lambda_{\min})^2){\sigma_{\max}}^2 \leq 4\lambda_{\min}^2\lambda_{\max}^2\\ \Longrightarrow\quad &((\lambda_{\max}+\lambda_{\min})^2+4(\lambda_{\max}+\lambda_{\min})\sqrt{\lambda_{\max}\lambda_{\min}}+4\lambda_{\max}\lambda_{\min}\\ &-(\lambda_{\max}+\lambda_{\min})^2){\sigma_{\max}}^2 \leq 4\lambda_{\min}^2\lambda_{\max}^2\\ \Longrightarrow\quad &{\sigma_{\max}}^2 \leq \frac{\lambda_{\min}^2\lambda_{\max}^2}{(\lambda_{\max}+\lambda_{\min})\sqrt{\lambda_{\max}\lambda_{\min}}+\lambda_{\max}\lambda_{\min}}. \qquad \square\end{aligned}$$

Table I. Spectral radius of the iteration matrix of the LHSS method ($n = 8$).

Difference scheme	q	α^*	$\rho(\alpha^*)$	α^t	$\rho(\alpha^t)$
Centered	1	0.7019	0.3609	2.1388	0.0989
Centered	10	0.7019	0.8735	2.1388	0.8411
Centered	100	0.7019	0.9240	0.1000	0.9891
Centered	1000	0.7019	0.9392	0.1000	0.9907
Upwind	1	0.7409	0.3449	2.1388	0.1010
Upwind	10	1.0918	0.8413	4.1776	0.6538
Upwind	100	4.6011	0.8908	8.2551	0.7971
Upwind	1000	39.6945	0.8951	75.5347	0.7948

Now we point out some errors in the section, containing the numerical results. The matrix A used for the examples is obtained from discretization of the three-dimensional convection-diffusion equation

$$-(u_{xx} + u_{yy} + u_{zz}) + q(u_x + u_y + u_z) = f(x, y, z)$$

on the unit cube $\Omega = [0, 1] \times [0, 1] \times [0, 1]$, with constant coefficient q and subject to Dirichlet-type boundary conditions. Discretizing this equation with seven-point finite difference and assuming the numbers (n) of grid points in all three directions are the same, a positive-definite system of linear equations with the coefficient matrix A ($n^3 \times n^3$) is obtained. If we define $h = 1/(n + 1)$ as the step size, $r = (qh)/2$ is the mesh Reynolds number, for the centered difference scheme we have (see [1, 2])

$$\begin{aligned} \lambda_{\min}(H) &= 6(1 - \cos \pi h), \qquad \lambda_{\max}(H) = 6(1 + \cos \pi h), \\ \sigma_{\max}(H) &= 6r \cos \pi h. \end{aligned}$$

Therefore, we have

$$\alpha^* = \frac{2\lambda_{\max}\lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} = 6(1 - \cos^2(\pi h)).$$

Moreover, for the upwind difference scheme

$$\begin{aligned} \lambda_{\min}(H) &= 6(1 + r)(1 - \cos \pi h), \qquad \lambda_{\max}(H) = 6(1 + r)(1 + \cos \pi h), \\ \sigma_{\max}(H) &= 6r \cos \pi h. \end{aligned}$$

In this case we obtain

$$\alpha^* = \frac{2\lambda_{\max}\lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} = 6(1 + r)(1 - \cos^2(\pi h)).$$

The values of α^* presented in Table I in [2] are different from the values obtained from the above formulas. Our numerical results show that the values of α^t presented in Table I in [2] are also erroneous. In Table I we report the correct ones.

Table I shows that with increasing q, the spectral radius becomes drastically large. More precisely, substituting $\lambda_{\min}$, $\lambda_{\max}$ and $\sigma_{\max}$ for the centered difference scheme in (2) results in

$$q \leq t_h,$$

where

$$t_h = \frac{2\tan \pi h}{h}\sqrt{\frac{\sin \pi h}{2+\sin \pi h}}. \tag{3}$$

Therefore, if $q \leq t_h$, then $\delta(\alpha^*) \leq \gamma(\bar{\alpha})$. For $n = 8$, we have $h = 1/9$ and $t_{1/9} = 2.5036$. Therefore, for $n = 8$ the LHSS method may be a good choice only for $q \leq 2.5036$. Similarly, for the upwind difference scheme, Eq. (2) results in

$$q \leq s_h,$$

where

$$s_h = \frac{2t_h}{1 - ht_h}.$$

If $n = 8$, then $s_{1/9} = 6.9370$. Hence, for $n = 8$ the LHSS method may be a good choice only for $q \leq 6.9370$. Evidently, $t_h, s_h \to 0$ as $h \to 0$. This shows that for sufficiently small h the LHSS may be the method of choice only for very small q.

Figure 1, plotted in [2] is wrong. We present the corrected one in Figure 1. The corrected version of Figure 2 in [2] is also plotted in Figure 2. This Figures show that the LHSS method is suitable only for small values of q.

There are some errors in the rest of the numerical results presented in [2], and their corrected versions confirm that the LHSS method is suitable only for small values of q and we omitted them here.

3. ACKNOWLEDGEMENT

The authors would like to thank the anonymous referee and our editor professor Maya Neytcheva for their helpful comments and suggestions.

REFERENCES

1. Bai ZZ, Golub GH, Ng MK, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, SIAM Journal on Matrix Analysis and Applications 2003; 24:603-626.
2. Li L, Huang TZ, Liu XP, Modified Hermitian and skew-Hermitian splitting methods for non-Hermitian positive-definite linear systems, Numerical Linear Algebra with Application 2007; 14:217-235.

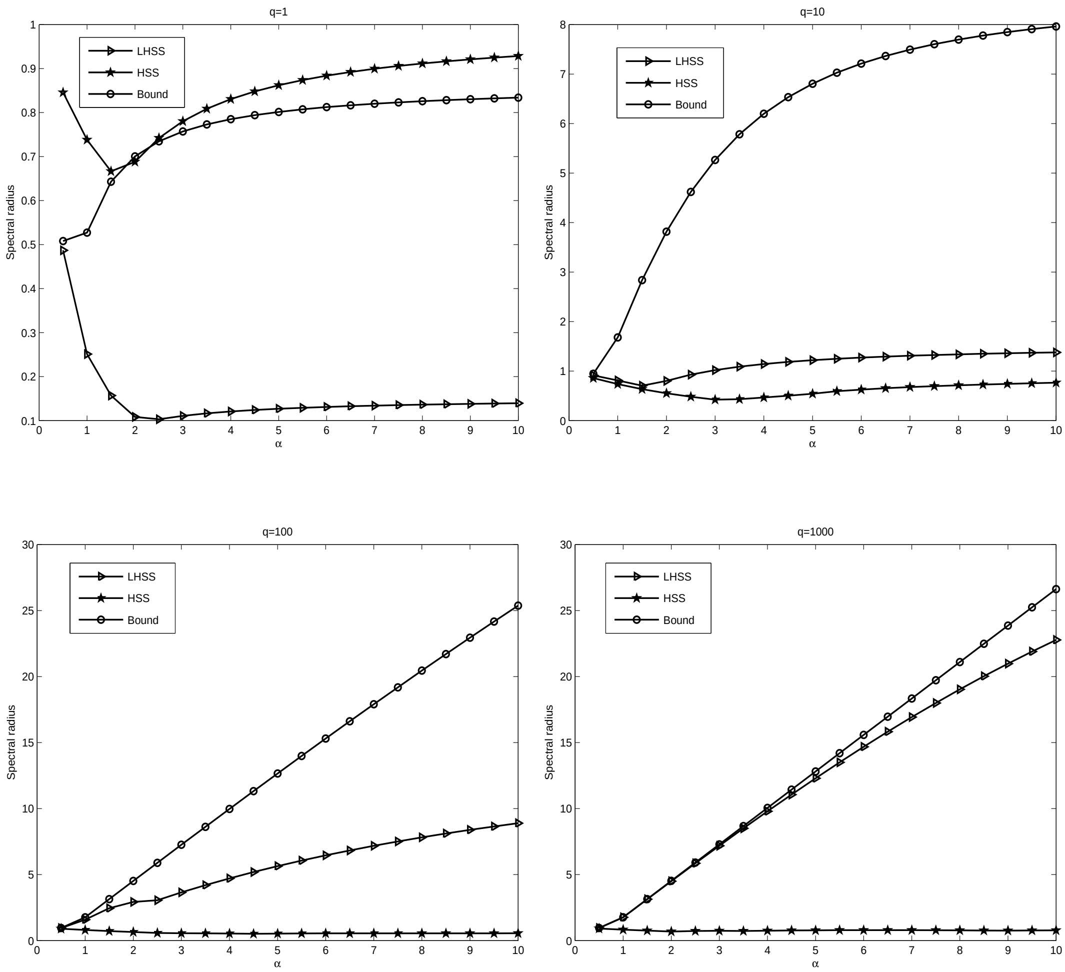

Figure 1. Centred difference scheme. The spectral radius $\rho(M(\alpha))$ of the iteration matrices of LHSS method and HSS method, and the bound $\delta(\alpha)$ with different α.

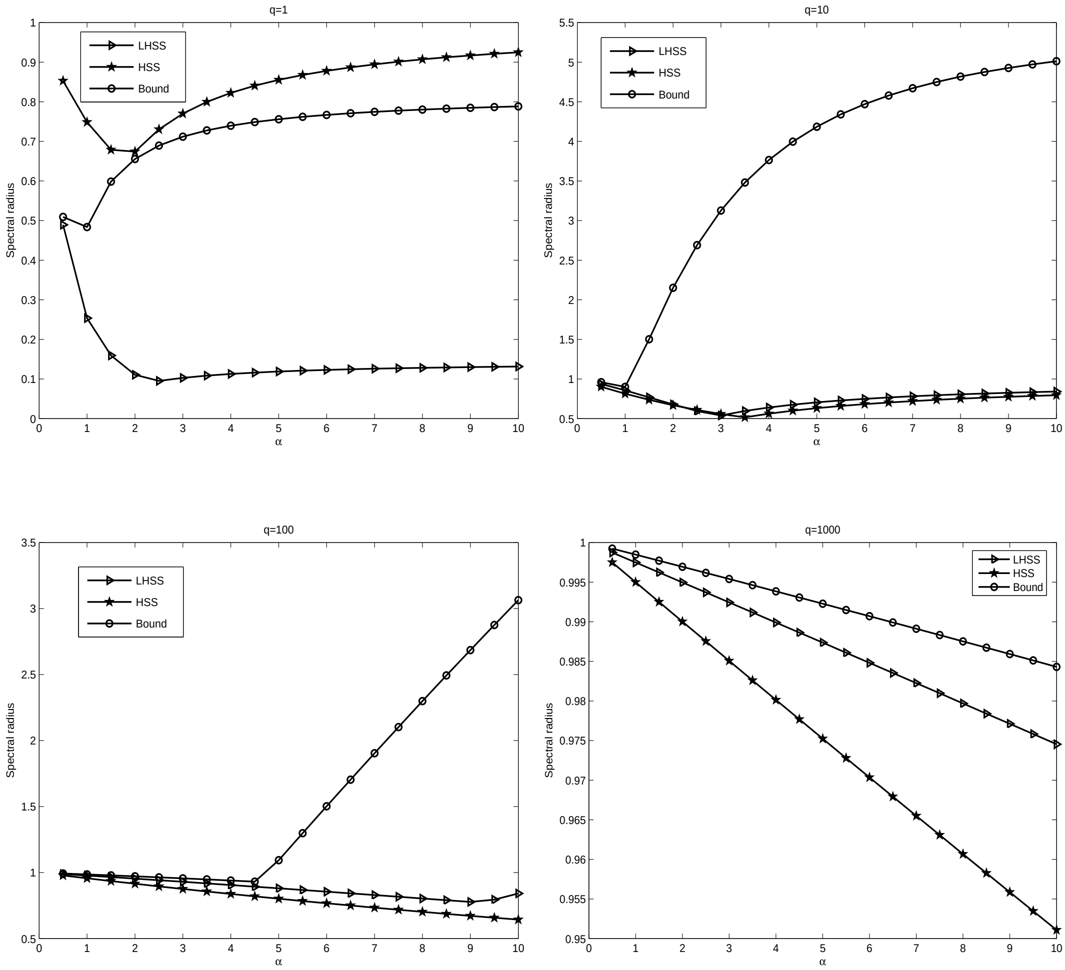

Figure 2. Upwind difference scheme. The spectral radius $\rho(M(\alpha))$ of the iteration matrices of LHSS method and HSS method, and the bound $\delta(\alpha)$ with different α.