On the solution of the fuzzy Sylvester matrix equation

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Abstract

In this paper, we consider the fuzzy Sylvester matrix equation $AX + XB = C$, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ are crisp M-matrices, C is an $n \times m$ fuzzy matrix and X is unknown. We first transform this system to an $(mn) \times (mn)$ fuzzy system of linear equations. Then we investigate the existence and uniqueness of a fuzzy solution to this system. We use the accelerated overrelaxation method to compute an approximate solution to this system. Some numerical experiments are given to illustrate the theoretical results.

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1 Introduction

Solution of a system of linear equations plays a crucial role in almost every field of science and engineering. In many applications, some of the system parameters are represented by fuzzy numbers rather than crisp numbers. Therefore, it is of importance to develop mathematical models and numerical procedures that would appropriately treat general fuzzy linear system of equations (hereafter it is denoted by FLSE). In [9], Friedman et al. considered a general model for solving a FLSE whose coefficient matrix is crisp and its right-hand side is an arbitrary fuzzy vector. They stated some conditions for the existence of a unique fuzzy solution to FLSE by using the embedding method [9] and converting the original system to a crisp linear system of equations. Then, many authors have studied FLSE. In [1], Allahviranloo used the Jacobi and Gauss-Seidel methods to solve FLSE. He also applied the successive overrelaxation iterative method for solving FLSE in [2]. In [8], Dehghan and Hashemi have applied several iterative methods for solving FLSE. In [13], Hashemi et al. have studied FLSE where the coefficient matrix is an crisp M-matrix. In [22], Wang and Zheng have studied some block iterative methods to solve FLSE.

It is well-known that the Sylvester matrix equation is of the form

$$AX + XB = C, \tag{1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{m\times m}$, $C = (c_{ij}) \in \mathbb{R}^{n\times m}$ and $X \in \mathbb{R}^{n\times m}$ is unknown. This matrix equation plays an important role in control theory, signal processing, filtering, model reduction, image restoration, decoupling techniques for ordinary and partial differential equations, implementation of implicit numerical methods for ordinary differential equations, and block-diagonalization of matrices (see, for example, [5, 6, 7, 14, 18] and the references therein). Standard solution methods for Sylvester equations of the form (1) are the Bartels-Stewart method [4] and the Hessenberg-Schur method [10]. The methods are based on transforming the coefficient matrices to Schur or Hessenberg form and then solving the corresponding linear system of equations directly by a backward substitution process. Therefore, these methods are classified as direct methods. Several iterative schemes to solve Sylvester equations have also been proposed; for methods focusing on large sparse systems see, e.g., [11, 15, 16, 20].

In this paper, we consider the Sylvester matrix equation (1) where the matrices A and B are crisp and the right-hand side matrix C is fuzzy. Then we investigate the existence of a fuzzy solution to this equation in the special case that A and B are M-matrices.

Throughout this paper, the vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ is said to be positive (nonnegative), if $x_i > 0$ ($x_i \geq 0$), $i = 1, 2, \ldots, n$. In this case, we write $x > 0$ ($x \geq 0$). Similar definitions can be written for matrices.

This paper is organized as follows. In section 2, some basic definitions and results on FLSE are given. Our main results concerning the existence of a fuzzy solution to the fuzzy Sylvester equation are given in section 3. In section 4, we discuss the application of the accelerated overrelaxation (AOR) method to compute an approximate solution to the fuzzy Sylvester equation. Section 5, is devoted to some numerical results illustrating the theoretical results. Some concluding remarks are given in section 6.

2 Preliminaries

In this section, we recall some of the basic notations of fuzzy number arithmetic and fuzzy linear system of equations.

Following [9], a fuzzy number in parametric form is defined an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfies the following requirements

1) $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0, 1]$,

2) $\overline{u}(r)$ is a bounded right continuous nonincreasing function over $[0, 1]$,

3) $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

A crisp number α is represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$. A popular representation for fuzzy number is the trapezoidal representation $u = (x_0, y_0, \alpha, \beta)$ with defuzzifier interval $[x_0, y_0]$, left fuzziness α and right fuzziness β [13]. The membership function of this trapezoidal number is as follows:

$$u(x) = \begin{cases} \frac{1}{\alpha}(x - x_0 + \alpha), & x_0 - \alpha \leq x \leq x_0, \\ 1, & x_0 \leq x \leq y_0, \\ \frac{1}{\beta}(y_0 - x + \beta), & y_0 \leq x \leq y_0 + \beta, \\ 0, & otherwise. \end{cases}$$

The parametric form of the number is

$$\underline{u}(r) = x_0 - \alpha + \alpha r, \quad \overline{u}(r) = y_0 + \beta - \beta r.$$

When $x_0 = y_0$, a trapezoidal fuzzy number is reduced to a triangular fuzzy number.

For introducing the FLSE and defining its solution, we recall the arithmetic operations of arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r))$ and $y = (\underline{y}(r), \overline{y}(r))$, $0 \leq r \leq 1$ and real number k as follows

a) $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,

b) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$,

c) $kx = \begin{cases} (k\underline{x}(r), k\overline{x}(r)), & k \geq 0, \\ (k\overline{x}(r), k\underline{x}(r)), & k < 0. \end{cases}$

Definition 2.1. ([9]) Consider the $n \times n$ linear system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = y_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = y_2, \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = y_n, \end{cases} \tag{2}$$

where the coefficient matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is crisp and $y = (y_1, y_2, \ldots, y_n)^T$ is a vector of fuzzy numbers. This system is called a FSLE.

Definition 2.2. ([9]) A fuzzy number vector $x = (x_1, x_2, \ldots, x_n)^T$ where $x_i = (\underline{x_i}(r), \overline{x_i}(r))$, $0 \leq r \leq 1$, $i = 1, 2, \ldots, n$, is called a solution of the fuzzy linear system of equations (2) if

$$\begin{cases} \underline{\sum_{j=1}^n a_{ij}x_j} = \sum_{j=1}^n \underline{a_{ij}x_j} = \underline{y_i}, \\ \overline{\sum_{j=1}^n a_{ij}x_j} = \sum_{j=1}^n \overline{a_{ij}x_j} = \overline{y_i}, \end{cases} \quad i = 1, 2, \ldots, n. \tag{3}$$

If, for a particular i, $a_{ij} > 0$, $j = 1, 2, \ldots, n$, we simply get

$$\sum_{j=1}^{n} a_{ij}\underline{x}_j = \underline{y}_i, \qquad \sum_{j=1}^{n} a_{ij}\overline{x}_j = \overline{y}_i.$$

In general, an arbitrary equation for either $\underline{y}_i$ or $\overline{y}_i$ is a linear combination of $\underline{x}_j$'s and $\overline{x}_j$'s. Therefore, in order to solve Eq. (2) one must solve a $(2n) \times (2n)$ crisp linear system of equations where the right-hand side vector is the function vector

$$Y = \begin{pmatrix} \underline{Y} \\ \overline{Y} \end{pmatrix},$$

where $\underline{Y} = (\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_n)^T$ and $\overline{Y} = (\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n)^T$. We rearrange the system (3) in the following form

$$\begin{cases} s_{11}\underline{x}_1 + s_{12}\underline{x}_2 + \ldots + s_{1n}\underline{x}_n - s_{1,n+1}\overline{x}_1 - s_{1,n+2}\overline{x}_2 - \ldots - s_{1,2n}\overline{x}_n = \underline{y}_1, \\ \qquad\vdots \\ s_{n1}\underline{x}_1 + s_{n2}\underline{x}_2 + \ldots + s_{nn}\underline{x}_n - s_{n,n+1}\overline{x}_1 - s_{n,n+2}\overline{x}_2 - \ldots - s_{n,2n}\overline{x}_n = \underline{y}_n, \\ -s_{n+1,1}\underline{x}_1 - s_{n+1,2}\underline{x}_2 - \ldots - s_{n+1,n}\underline{x}_n + s_{n+1,n+1}\overline{x}_1 + s_{n+1,n+2}\overline{x}_2 + \ldots + s_{n+1,2n}\overline{x}_n = \overline{y}_1, \\ \qquad\vdots \\ -s_{2n,1}\underline{x}_1 - s_{2n,2}\underline{x}_2 - \ldots - s_{2n,n}\underline{x}_n + s_{n+1,n+1}\overline{x}_1 + s_{2n,n+2}\overline{x}_2 + \ldots + s_{2n,2n}\overline{x}_n = \overline{y}_n, \end{cases} \tag{4}$$

where s_{ij} are determined as follows

$$\begin{cases} a_{ij} \geq 0 \Rightarrow s_{ij} = a_{ij}, \ s_{i+n,j} = a_{ij}, \\ a_{ij} < 0 \Rightarrow s_{i,j+n} = -a_{ij}, \ s_{i+n,j} = -a_{ij}, \end{cases} \tag{5}$$

and any a_{ij} which is not determined by Eq. (5) is zero. Using the matrix notation we obtain the $(2n) \times (2n)$ linear system of equations

$$SX = Y, \tag{6}$$

where

$$X = \begin{pmatrix} \underline{X} \\ \overline{X} \end{pmatrix},$$

in which $\underline{X} = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)^T$ and $\overline{X} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T$. It is easy to see that Eq. (6) is equivalent to

$$\begin{pmatrix} S_1 & -S_2 \\ -S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{X} \\ \overline{X} \end{pmatrix} = \begin{pmatrix} \underline{Y} \\ \overline{Y} \end{pmatrix}, \tag{7}$$

where $S_1, S_2 \geq 0$ and $A = S_1 - S_2$. In other words, S_1 contains the positive entries of A, S_2 the absolute values of the negative entries of A and $A = S_1 - S_2$. To our knowledge, the above rearrangement was first introduced by Allahviranloo in [1].

Remark 2.1. In [9], the authors considered another rearrangement which results in the following linear system of equations

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{X} \\ -\overline{X} \end{pmatrix} = \begin{pmatrix} \underline{Y} \\ -\overline{Y} \end{pmatrix}, \tag{8}$$

where S_1 and S_2 are as before.

We will see that Eq. (7) has some advantages over Eq. (8).

Theorem 2.1. *The coefficient matrix in Eq. (7) is nonsingular if and only if $A = S_1 - S_2$ and $S_1 + S_2$ are both nonsingular.*

Proof. The proof is similar to Theorem 1 in [9]. □

Theorem 2.2. *The components of the unique solution X of Eq. (7) represent a solution fuzzy vector to the system (2) for arbitrary Y if and only if S^{-1} is nonnegative, i.e., $S^{-1} \geq 0$.*

Proof. The proof is similar to Theorem 3 in [9]. □

Definition 2.3. Let $X = \{(\underline{x}_i(r), \overline{x}_i(r)), 1 \leq i \leq n\}$ be a solution of $SX = Y$. The fuzzy number vector $u = \{(\underline{u}_i(r), \overline{u}_i(r)), 1 \leq i \leq n\}$ defined by

$$\begin{aligned} \underline{u}_i(r) &= \min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\} \\ \overline{u}_i(r) &= \max\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\} \end{aligned}$$

is called the fuzzy solution of $SX = Y$. If $(\underline{x}_i(r), \overline{x}_i(r))$, $i = 1, 2, \ldots, n$, are all fuzzy numbers and $\underline{u}_i(r) = \underline{x}_i(r)$, $\overline{u}_i(r) = \overline{x}_i(r)$, $1 \leq i \leq n$, then u is called a strong fuzzy solution. Otherwise, u is a weak fuzzy solution.

3 Main results

In this section, we consider the fuzzy Sylvester equation (FSE), i.e., the matrix equation (1) where A and B are real and C is a fuzzy matrix and verify the existence of a fuzzy solution for it.

Definition 3.1. ([17]) Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$. Then the Kronecker product of A and B is defined as the matrix $A \otimes B = (a_{ij}B) \in \mathbb{R}^{mp \times nq}$.

Theorem 3.1. Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{r\times s}$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^s$. Then

$$(A \otimes B)(x \otimes y) = Ax \otimes By.$$

Proof. It follows immediately from Theorem 13.3 in [17]. $\square$

Definition 3.2. ([17]) Let $z_i \in \mathbb{R}^n$, $i = 1, \ldots, n$, denotes the ith column of $Z \in \mathbb{R}^{n\times m}$ so that $Z = (z_1, \ldots, z_n)$. Then $\mathrm{vec}(Z)$ is defined to be the (mn)-vector defined as

$$\mathrm{vec}(Z) = \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix}.$$

Using definition 3.1, the Sylvester matrix equation (1) can be written in the form

$$\mathcal{A}\mathcal{X} = \mathcal{C}, \tag{9}$$

where

$$\begin{aligned} \mathcal{A} &= (I_m \otimes A) + (B^T \otimes I_n) \in \mathbb{R}^{(mn)\times(mn)}, \\ \mathcal{X} &= \mathrm{vec}(X), \\ \mathcal{C} &= \mathrm{vec}(C), \end{aligned}$$

in which I_m and I_n are the identity matrices of orders m and n, respectively (see for example [17]). There exists a unique solution to (9) if and only if $\mathcal{A}$ is nonsingular.

Definition 3.3. ([19]) A matrix $A = (a_{ij})$ is said to be an M-matrix if $a_{ii} > 0$ for $i = 1, \ldots, n$, $a_{ij} \le 0$, for $i \ne j$, A is nonsingular and $A^{-1} \ge 0$.

Theorem 3.2. ([3, Lemma 6.4]) *A matrix $A = (a_{ij})$ with $a_{ij} \le 0$, $i \ne j$, is an M-matrix if and only if there exists a positive vector x, such that Ax is positive.*

Now, we state and prove the following theorem.

Theorem 3.3. *Let A and B be two M-matrices. Then the matrix $\mathcal{A}$ is also an M-matrix.*

Proof. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two M-matrices. It is easy to see that the matrix $\mathcal{A}$ is of the form [17]

$$\mathcal{A} = \begin{pmatrix} A + b_{11}I_n & b_{21}I_n & \cdots & b_{m1}I_n \\ b_{12}I_n & A + b_{22}I_n & \cdots & b_{m2}I_n \\ \vdots & \vdots & \ddots & \vdots \\ b_{1m}I_n & b_{2m}I_n & \cdots & A + b_{mm}I_n \end{pmatrix}.$$

This shows that the off-diagonal entries of $\mathcal{A}$ are nonpositive. Obviously, B^T is an M-matrix. Therefore, by Theorem 3.2 there exist two positive vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, such that Ax and $B^T y$ are positive. Let $v = y \otimes x \in \mathbb{R}^{mn}$. Evidently, v is positive. Now, by Theorem 3.1 we obtain

$$\begin{aligned} \mathcal{A}v &= [(I_m \otimes A) + (B^T \otimes I_n)](y \otimes x) \\ &= (I_m \otimes A)(y \otimes x) + (B^T \otimes I_n)(y \otimes x) \\ &= I_m y \otimes Ax + B^T y \otimes I_n x \\ &= y \otimes Ax + B^T y \otimes x \\ &> 0. \end{aligned}$$

This shows that $\mathcal{A}$ is an M-matrix. □

This theorem shows that Eq. (9) is a linear system of equations with a crisp coefficient M-matrix and fuzzy right-hand side vector. Therefore, the existence of a fuzzy vector $\mathcal{X}$ for Eq. (9) should be investigated. Similar to the method presented in [13], one may extend this system to a $(2mn) \times (2mn)$ crisp linear system of equations as (8). In [13], the authors showed that the coefficient matrix of the extended linear system is an H-matrix and proposed a method to solve it by using the Schur complement of the coefficient matrix. In continuation, we consider an extension of the form (7) for Eq. (9) and show that the coefficient matrix of the extended system is an M-matrix.

We first extend the system (9) to $(2mn) \times (2mn)$ of the form (7). Since $\mathcal{A} = (\alpha_{ij})$ is an M-matrix, we have $\alpha_{ii} > 0$, $i = 1, 2, \ldots, n$, and $\alpha_{ij} \leq 0$, $i \neq j$. Therefore we have

$$S_1 = \mathrm{diag}(\mathcal{A}) = \mathcal{D}, \quad S_2 = \mathcal{D} - \mathcal{A}.$$

Hence, the extended linear system of equations can be written as

$$\mathcal{S}\mathfrak{X} = \mathfrak{C}, \tag{10}$$

where

$$\mathcal{S} = \begin{pmatrix} \mathcal{D} & \mathcal{A} - \mathcal{D} \\ \mathcal{A} - \mathcal{D} & \mathcal{D} \end{pmatrix}, \quad \mathfrak{X} = \begin{pmatrix} \underline{\mathcal{X}} \\ \overline{\mathcal{X}} \end{pmatrix}, \quad \text{and} \quad \mathfrak{C} = \begin{pmatrix} \underline{\mathcal{C}} \\ \overline{\mathcal{C}} \end{pmatrix}.$$

Theorem 3.3. *If $\mathcal{A}$ is an M-matrix, then the coefficient matrix of (10), $\mathcal{S}$, is also an M-matrix.*

Proof. Since $\mathcal{A}$ is an M-matrix, there exists a positive vector x, such that $\mathcal{A}x > 0$. Let $y = (x^T, x^T)^T$. Obviously $y > 0$. Now we have

$$\begin{aligned} \mathcal{S}y &= \begin{pmatrix} \mathcal{D} & \mathcal{A}-\mathcal{D} \\ \mathcal{A}-\mathcal{D} & \mathcal{D} \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{A}x \\ \mathcal{A}x \end{pmatrix} \\ &> 0. \end{aligned}$$

Evidently, off-diagonal entries of $\mathcal{S}$ are nonpositive. Therefore, by Theorem 3.2, $\mathcal{S}$ is an M-matrix. □

Corollary 3.1. *The unique solution of Eq. (10) is a fuzzy vector.*

Proof. From Theorem 3.3 the coefficient matrix of Eq. (10), $\mathcal{S}$, is an M-matrix. Therefore, $\mathcal{S}^{-1} \geq 0$. Hence, from Theorem 2.2 the solution of the Eq. (10) is a fuzzy vector. □

Remark 3.1. From Corollary 3.1, we observe that, if A is M-matrix then the coefficient matrix of the system (7) is also an M-matrix. Since the inverse of an M-matrix is nonnegative, by Theorem 2.2 we see that the solution of Eq. (7) is a fuzzy vector. Now consider the system (8). If A is an M-matrix, then the coefficient matrix of this system is an H-matrix (see [13, Theorem 3.6]). As we know, the inverse of and H-matrix is not necessary nonnegative. Therefore, the solution of Eq. (8) is not necssary a fuzzy vector (see example 1 in [13]).

4 The AOR method to solve Eq. (10)

System (10) can be solved by direct methods such as the method presented by Hashemi et al. in [13] which is based on the Schur complement of the coefficient matrix. In this section, we use the AOR method [12] to solve (10). We split A into $A = D - L - U$, where D, $-L$ and $-U$ are the diagonal, strictly lower and strictly upper triangular parts of A, respectively. In this case, the AOR method to solve $Ax = b$, can be written as

$$x^{(k+1)} = \mathcal{L}_{\gamma,\omega} x^{(k)} + \omega(D - \gamma L)^{-1} b, \quad k = 0, 1, 2, \ldots,$$

where

$$\mathcal{L}_{\gamma,\omega} = (D - \gamma L)^{-1}[(1-\omega)D + (\omega - \gamma)L + \omega U],$$

in which γ and $\omega \neq 0$ are two real parameters. As it is well-known, for convenient values of γ and ω, the AOR method becomes the well-known iterative methods:

$$\begin{array}{ll} \gamma = 0,\ \omega = 1 & \text{Jacobi Method,} \\ \gamma = 1,\ \omega = 1 & \text{Gauss-Seidel Method,} \\ \gamma = \omega & \text{Successive Overrelaxation (SOR) Method.} \end{array}$$

It is also well-known that the AOR method (as a result Jacobi, Gauss-Seidel and SOR methods) is convergent for M-matrices provided that $0 \leq \gamma \leq \omega \leq 1$ and $\omega \neq 0$ (see [23, Corollary 2.3]). Therefore we use the AOR method to solve system (10). It is straightforward to verify that the AOR method to solve Eq. (10) can be written as Algorithm 1.

`Algorithm 1: AOR method for Eq. (10)`

1. `Choose initial guesses` $\underline{\mathcal{X}}^{(0)}$ `and` $\overline{\mathcal{X}}^{(0)}$ `and set` $\mathcal{P} = \mathcal{D}^{-1}\mathcal{A} - I$
2. `For` $k = 0, 1, \ldots,$ `Do`
3. $\quad\underline{\mathcal{X}}^{(k+1)} = (1-\omega)\underline{\mathcal{X}}^{(k)} - \omega\mathcal{P}\overline{\mathcal{X}}^{(k)} + \omega\mathcal{D}^{-1}\underline{\mathcal{C}}$
4. $\quad\overline{\mathcal{X}}^{(k+1)} = \omega(\gamma-1)\mathcal{P}\underline{\mathcal{X}}^{(k)} + [(1-\omega)I + \gamma\omega\mathcal{P}^2]\overline{\mathcal{X}}^{(k)} - \omega\gamma\mathcal{P}\mathcal{D}^{-1}\underline{\mathcal{C}} + \omega\mathcal{D}^{-1}\overline{\mathcal{C}}$
5. `If the stopping criterion satisfied, then stop`
6. `EndDo`

We mention that, in this algorithm, the matrix $\mathcal{D}$ is diagonal and its inverse can be computed easily.

5 Numerical examples

In this section, some numerical experiments are given to illustrate the theoretical results presented in this paper. The triangular fuzzy numbers are used in all of the following numerical examples. For a triangular fuzzy number $u = (\underline{u}(r), \overline{u}(r)) = (a + br, c + dr)$, we define its norm as (see [22])

$$\|x\| = \max\{|a|, |b|, |c|, |d|\}.$$

Therefore, the norm of

$$E^{(k+1)} = \begin{pmatrix} \underline{\mathcal{X}}^{(k+1)} - \underline{\mathcal{X}}^{(k)} \\ \overline{\mathcal{X}}^{(k+1)} - \overline{\mathcal{X}}^{(k)} \end{pmatrix} = \begin{pmatrix} a_1 + b_1 r \\ a_2 + b_2 r \\ \vdots \\ a_{4mn} + b_{4mn} r \end{pmatrix}, \tag{11}$$

where a_i's and b_i's are crisp numbers, are defined as

$$\|E^{(k+1)}\| = \max_i \{|a_i|, |b_i|\}.$$

We use

$$\|E^{(k+1)}\| < 10^{-4},$$

as the stopping criterion in Algorithm 1 and a zero vector is always used as the initial guess.

Example 1. In this example, we consider the fuzzy Sylvester equation (1) with

$$A = \begin{pmatrix} 3 & -3 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -2 \\ -3 & 4 \end{pmatrix},$$

and fuzzy right hand side

$$C = \begin{pmatrix} (-21+11r, 4-14r) & (19r, 31-12r) \\ (-1+8r, 15-8r) & (-16+9r, 3-10r) \end{pmatrix}.$$

It is easy to see that A and B are M-matrices. The exact solution of this system is

$$X^* = \begin{pmatrix} (r, 2-r) & (1+2r, 4-r) \\ (1+r, 3-r) & (-1+r, 1-r) \end{pmatrix}.$$

The coefficient and the right-hand side of the system $\mathcal{A}\mathcal{X} = \mathcal{C}$ are as following

$$\mathcal{A} = \begin{pmatrix} 5 & -3 & -3 & 0 \\ -1 & 4 & 0 & -3 \\ -2 & 0 & 7 & -3 \\ 0 & -2 & -1 & 6 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} (-21+11r, 4-14r) \\ (-1+8r, 15-8r) \\ (19r, 31-12r) \\ (-16+9r, 3-10r) \end{pmatrix}.$$

It can be easily verified that the matrix $\mathcal{A}$ is an M-matrix. Now, the coefficient and the right hand side of the system $\mathcal{S}\mathfrak{X} = \mathfrak{C}$ may be written as

$$\mathcal{S} = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & -3 & -3 & 0 \\ 0 & 4 & 0 & 0 & -1 & 0 & 0 & -3 \\ 0 & 0 & 7 & 0 & -2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 6 & 0 & -2 & -1 & 0 \\ 0 & -3 & -3 & 0 & 5 & 0 & 0 & 0 \\ -1 & 0 & 0 & -3 & 0 & 4 & 0 & 0 \\ -2 & 0 & 0 & -3 & 0 & 0 & 7 & 0 \\ 0 & -2 & -1 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}, \quad \mathfrak{C} = \begin{pmatrix} -21+11r \\ -1+8r \\ 19r \\ -16+9r \\ 4-14r \\ 15-8r \\ 31-12r \\ 3-10r \end{pmatrix}.$$

Algorithm 1 with $\omega = 0.9$ and $\gamma = 0.8$ to solve $\mathcal{S}\mathfrak{X} = \mathfrak{C}$ converges in 25 iterations and gives the following approximate solution to the Sylvester equation:

$$X^{(25)} = \begin{pmatrix} (-0.0010 + 1.0002r, 1.9995 - 0.9998r) & (0.9997 + 2.0001r, 3.9995 - 0.9999r) \\ (0.9996 + 1.0002r, 2.9994 - 0.9999r) & (-1.0004 + 1.0001r, 0.9998 - 0.9999r) \end{pmatrix}.$$

As we observe there is a very good agreement between the approximate solution $X^{(25)}$ and the exaction solution X^*. Moreover, the provided solution is a strong fuzzy solution.

Example 2. Our second example is devoted to the fuzzy Sylvester equation with

$$A = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -5 \\ -3 & 5 \end{pmatrix}.$$

and fuzzy right-hand side

$$C = \begin{pmatrix} (-25 + 16r, 10 - 19r) & (-28 + 26r, 14 - 16r) \\ (-15 + 22r, 26 - 19r) & (-18 + 20r, 31 - 29r) \\ (-10 + 17r, 24 - 17r) & (-4 + 20r, 35 - 19r) \end{pmatrix}.$$

The exact solution of the corresponding Sylvester equation is

$$X^* = \begin{pmatrix} (1 + r, 3 - r) & (1 + 2r, 4 - r) \\ (1 + 2r, 5 - 2r) & (2 + r, 5 - 2r) \\ (2 + r, 4 - r) & (3 + r, 5 - r) \end{pmatrix}.$$

For this problem the coefficient and right-hand side of the system $\mathcal{A}\mathcal{X} = \mathcal{C}$ are as follows

$$\mathcal{A} = \begin{pmatrix} 6 & -3 & -1 & -3 & 0 & 0 \\ -1 & 7 & -1 & 0 & -3 & 0 \\ -1 & -2 & 9 & 0 & 0 & -3 \\ -5 & 0 & 0 & 7 & -3 & -1 \\ 0 & -5 & 0 & -1 & 8 & -1 \\ 0 & 0 & -5 & -1 & -2 & 10 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} (-25 + 16r, 10 - 19r) \\ (-15 + 22r, 26 - 19r) \\ (-10 + 17r, 24 - 17r) \\ (-28 + 26r, 14 - 16r) \\ (-18 + 20r, 31 - 29r) \\ (-4 + 20r, 35 - 19r) \end{pmatrix}.$$

As we mentioned $\mathcal{A}\mathcal{X} = \mathcal{C}$ is a fuzzy linear system of equations with coefficient M-matrix. Then, we obtain the coefficient and right hand side of $\mathcal{S}\mathfrak{X} = \mathfrak{C}$ as follows

$$\mathcal{S} = \begin{pmatrix} \mathcal{D} & \mathcal{P} \\ \mathcal{P} & \mathcal{D} \end{pmatrix},$$

where

$$\mathcal{D} = \begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 0 & -3 & -1 & -3 & 0 & 0 \\ -1 & 0 & -1 & 0 & -3 & 0 \\ -1 & -2 & 0 & 0 & 0 & -3 \\ -5 & 0 & 0 & 0 & -3 & -1 \\ 0 & -5 & 0 & -1 & 0 & -1 \\ 0 & 0 & -5 & -1 & -2 & 0 \end{pmatrix},$$

and

$$\mathfrak{C} = \begin{pmatrix} \underline{\mathcal{C}} \\ \overline{\mathcal{C}} \end{pmatrix}, \quad \text{where} \quad \underline{\mathcal{C}} = \begin{pmatrix} -25+16r \\ -15+22r \\ -10+17r \\ -28+26r \\ -18+20r \\ -4+20r \end{pmatrix}, \quad \text{and} \quad \overline{\mathcal{C}} = \begin{pmatrix} 10-19r \\ 26-19r \\ 24-17r \\ 14-16r \\ 31-29r \\ 35-19r \end{pmatrix}.$$

Algorithm 1 with $\omega = 0.9$ and $\gamma = 0.8$ to solve $\mathcal{S}\mathfrak{X} = \mathfrak{C}$ converges in 52 iterations and gives the following approximate solution to FSE:

$$X^{(52)} = \begin{pmatrix} (0.9982+1.0004r, 2.9983-0.9996r) & (0.9979+2.0005r, 3.9981-0.9995r) \\ (0.9991+2.0002r, 4.9992-1.9998r) & (1.9990+1.0003r, 4.9990-1.9998r) \\ (1.9992+1.0002r, 3.9993-0.9998r) & (2.9991+1.0002r, 4.9992-0.9998r) \end{pmatrix}.$$

As we see the AOR method has provided a good approximate solution to FSE. In addition, the computed solution is a strong fuzzy solution.

Example 3. In this example, we consider the FSE with

$$A = \begin{pmatrix} -3 & -2 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & -1 \\ -3 & 5 & -1 \\ -2 & -3 & 4 \end{pmatrix}.$$

and fuzzy right-hand side

$$C = \begin{pmatrix} (-24+26r, 19-17r) & (-26+23r, 13-16r) & (-1+12r, 38-27r) \\ (-21+13r, 3-11r) & (-4+11r, 24-17r) & (-10+11r, 9-8r) \end{pmatrix}.$$

The exact solution of the corresponding Sylvester equation is

$$X^{*} = \begin{pmatrix} (1+3r, 6-2r) & (1+r, 3-r) & (2+r, 6-3r) \\ (2+r, 4-r) & (2+r, 5-2r) & (1+r, 3-r) \end{pmatrix}.$$

For this problem the coefficient and right-hand side of the system $\mathcal{A}\mathcal{X} = \mathcal{C}$ are as follows

$$\mathcal{A} = \begin{pmatrix} 5 & -2 & -3 & 0 & -2 & 0 \\ -1 & 3 & 0 & -3 & 0 & -2 \\ -1 & 0 & 8 & -2 & -3 & 0 \\ 0 & -1 & -1 & 6 & 0 & -3 \\ -1 & 0 & -1 & 0 & 7 & -2 \\ 0 & -1 & 0 & -1 & -1 & 5 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} (-24+26r, 19-17r) \\ (-21+13r, 3-11r) \\ (-26+23r, 13-16r) \\ (-4+11r, 24-17r) \\ (-1+12r, 38-27r) \\ (-10+11r, 9-8r) \end{pmatrix}.$$

The coefficient of the system $\mathcal{A}\mathcal{X}=\mathcal{C}$ is an M-matrix and the coefficient and right hand side of $\mathcal{S}\mathfrak{X}=\mathfrak{C}$ can be written as

$$\mathcal{S}=\begin{pmatrix} \mathcal{D} & \mathcal{P} \\ \mathcal{P} & \mathcal{D} \end{pmatrix},$$

where

$$\mathcal{D}=\begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}, \quad \mathcal{P}=\begin{pmatrix} 0 & -2 & -3 & 0 & -2 & 0 \\ -1 & 0 & 0 & -3 & 0 & -2 \\ -1 & 0 & 0 & -2 & -3 & 0 \\ 0 & -1 & -1 & 0 & 0 & -3 \\ -1 & 0 & -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & -1 & -1 & 0 \end{pmatrix},$$

and

$$\mathfrak{C}=\begin{pmatrix} \underline{\mathcal{C}} \\ \overline{\mathcal{C}} \end{pmatrix}, \quad \text{where} \quad \underline{\mathcal{C}}=\begin{pmatrix} -24+26r \\ -21+13r \\ -26+23r \\ -4+11r \\ -1+12r \\ -10+11r \end{pmatrix}, \quad \text{and} \quad \overline{\mathcal{C}}=\begin{pmatrix} 19-17r \\ 3-11r \\ 13-16r \\ 24-17r \\ 38-27r \\ 9-8r \end{pmatrix}.$$

Algorithm 1 with $\omega=0.75$ and $\gamma=0.6$ to solve $\mathcal{S}\mathfrak{X}=\mathfrak{C}$ converges in 99 iterations and gives the following approximate solution to the Sylvester equation:

$$X^{(99)}=\begin{pmatrix} (0.9958+3.0010r, 5.9959-1.9991r) & (0.9982+1.0004r, 2.9982-0.9996r) \\ (1.9945+1.0013r, 3.9946-0.9988r) & (1.9976+1.0006r, 4.9977-1.9995r) \end{pmatrix.$$

$$\begin{matrix} (1.9985+1.0004r, 5.9985-2.9997r) \\ (0.9980+1.0005r, 2.9980-0.9995r) \end{pmatrix}.$$

As we observe, Algorithm 1 has given a good approximate solution to FSE. Moreover, the computed solution is a strong fuzzy solution.

Example 4. As we mentioned in section 1, one of the application of FSE is in block diagonalization of matrices. Consider the block fuzzy matrix

$$H=\begin{pmatrix} B & 0 \\ C & -A \end{pmatrix}$$

where

$$A=\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad B=\begin{pmatrix} 3 & -1 \\ -2 & 3 \end{pmatrix},$$

and

$$C=\begin{pmatrix} (-6+14r,10-8r) & (-8+7r,7-8r) \\ (4+9r,-1-16r) & (-6+9r,5-8r) \end{pmatrix}.$$

Let X be the solution of FSE $AX+XB=C$ and

$$W=\begin{pmatrix} I_2 & 0 \\ X & I_2 \end{pmatrix},$$

where I_2 is the identity matrix of order 2. It is easy to verify that the matrix W can be used to block-diagonalize the matrix H (see [6]). In fact, we have

$$W^{-1}HW=\begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}.$$

Therefore, to compute X, we should solve the FSE $AX+XB=C$. Similar to the previous examples, Algorithm 1 with $\omega=0.6$ and $\gamma=0.5$, in 21 iterations, gives the following approximate solution

$$X^{(21)}=\begin{pmatrix} (-0.9999+2.0002r,1.9996-0.9999r) & (-1.0003+1.0001r,1.0000-0.9999r) \\ (1.9993+1.0002r,-0.9999-1.9998r) & (-0.9999+1.0002r,0.9996-0.9999r) \end{pmatrix}.$$

Here we mention that the exact solution is:

$$X^{*}=\begin{pmatrix} (-1+2r,2-r) & (-1+r,1-r) \\ (2+r,-1-2r) & (-1+r,1-r) \end{pmatrix}.$$

We observe that there is a very good agreement between the approximate solution $X^{(21)}$ and the exact solution X^*.

6 Conclusion

We have considered the Sylvester matrix equation $AX+XB=C$, where A and B are M-matrices and C is an arbitrary fuzzy matrix. We then transform this equation to a fuzzy linear system of equations where the coefficient matrix is crisp and the right-hand of the system is a fuzzy vector. We have shown that the coefficient matrix of this system is an M-matrix. Next, we extend this system to a crisp system of linear equations with coefficient M-matrix. The existence and uniqueness of a fuzzy matrix solution to the fuzzy Sylvester equation has also been discussed. We applied the accelerated overrelaxation method to compute an approximate solution to the fuzzy Sylvester equation. Some numerical results have been given to illustrate the theoretical results presented in section 3. The presented numerical results support the theoretical results. Meanwhile, we have given some comments on the paper [13].

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