On a class of multi-level preconditioners for Z-matrices

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Abstract

In this paper, we study the convergence of a class of multi-level preconditioners and corresponding block AOR iterative methods. Then, we give some comparison theorems for different preconditioning levels. Finally, the effectiveness of the proposed methods are shown by some numerical experiments.

AMS Subject Classification : 65F10.
Keywords: System of linear equations, Block preconditioner, Block AOR iterative method, Z-matrix, M-matrix.

1. Introduction

Consider the system of linear equations

$$Ax = b, \qquad A \in \mathbb{R}^{n\times n}, \quad x, b \in \mathbb{R}^n \tag{1.1}$$

where A is nonsingular matrix blocked in the form

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1p} \\ A_{21} & A_{22} & \dots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \dots & A_{pp} \end{pmatrix}.$$

This type of matrices typically arise in computational fluid dynamics, circuit simulation, thermal and structural problems. It is assumed that the diagonal blocks A_{ii} of A are square matrices with the same order. A stationary iterative method to solve Eq. (1.1) can be written as

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \quad k = 0, 1, 2, \dots,$$

in which $A = M - N$, where $M, N \in \mathbb{R}^{n\times n}$ and M is nonsingular. It is well-known that this iterative method for each x_0 converges to the unique solution of Eq. (1.1) if and only if $\rho(M^{-1}N) < 1$, where $\rho(M^{-1}N)$ is the spectral radius of $M^{-1}N$. Moreover, the smaller $\rho(M^{-1}N)$ is, the faster the iteration converges. Let A_{ii}, $i = 1, 2, \dots, p$, are nonsingular. Without loss of generality, we can assume that A can be partitioned into $p \times p$ block matrix form

$$A = \begin{pmatrix} I_{11} & A_{12} & \dots & A_{1p} \\ A_{21} & I_{22} & \dots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \dots & I_{pp} \end{pmatrix}.$$

In this case, we split A into

$$A = I - L - U, \tag{1.2}$$

where

$$L = (L_{ij}) = \begin{cases} -A_{ij}, & j < i, \\ 0, & j \geq i, \end{cases} \qquad U = (U_{ij}) = \begin{cases} -A_{ij}, & j > i, \\ 0, & j \leq i, \end{cases}$$

are block matrices consisting of the strictly block lower triangular, strictly block upper triangular parts of A, respectively, and $I_{ii} \in \mathbb{R}^{n_i \times n_i}$ is the identity matrix, $i = 1, 2, \ldots, p$, where $n_1 + n_2 + \cdots + n_p = n$. The block accelerated overrelaxation (BAOR) iterative method to solve Eq. (1.1) is defined by (see [5])

$$x^{(k+1)} = T_{r,w} x^{(k)} + w(I - rL)^{-1} b, \quad k = 0, 1, 2, \ldots,$$

in which

$$T_{r,w} = (I - rL)^{-1}[(1 - w)I + (w - r)L + wU],$$

where $w \neq 0$ and r are real parameters. For certain values of the parameters w and r the BAOR iterative method results in the block Jacobi, block Gauss-Seidel and the block SOR methods. Moreover, this method is reduced to the point AOR, SOR, Gauss-Seidel and Jacobi methods respectively when $n_i = 1, i = 1, 2, \ldots, p$ (see [5]).

In order to improve the convergence rate of a basic iterative method, several preconditioned iterative methods have been proposed in [3,4,6-15]. The main idea of these preconditioned iterative methods is to transform the original system to the preconditioned form $PAx = Pb$, where the matrix P is called a preconditioner.

In this paper, we investigate the preconditioners of the form

$$P^{(i)} = \begin{pmatrix} I_{11} & \ldots & -\alpha A_{1,i+1} & \ldots & 0 & 0 \\ 0 & I_{22} & \ldots & -\alpha A_{2,i+2} & 0 & 0 \\ \ldots & \ldots & \ddots & \ldots & \ddots & 0 \\ 0 & 0 & 0 & I_{p-i,p-i} & \ldots & -\alpha A_{p-i,p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & I_{pp} \end{pmatrix},$$

and

$$P^{(-i)} = \begin{pmatrix} I_{11} & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \ldots & 0 & \ldots & 0 \\ -\alpha A_{i+1,1} & \ldots & I_{i+1,i+1} & 0 & \ldots & 0 \\ 0 & -\alpha A_{i+2,2} & \ldots & I_{i+2,i+2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \ddots & 0 \\ 0 & 0 & \ldots & -\alpha A_{p,p-i} & \ldots & I_{pp} \end{pmatrix},$$

for $i = 1, 2, \ldots, p - 1$, in which α is a nonnegative real number. In [13], Wu et al. have applied the preconditioner $P^{(1)}$ to the point AOR method. In this paper firstly, we apply the preconditioners $P^{(i)}$ and $P^{(-i)}$ to the block AOR iterative method and give some comparison theorems. Then a multi-level preconditioned block AOR method with preconditioners $P^{(i)}$ and $P^{(-i)}$ is presented and analyzed.

For convenience, some notations, definitions and preliminaries that will be used in the following parts are given below. A matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ is called nonnegative and denoted by $A \geq 0$ if $a_{ij} \geq 0$ for all i and j, and A is called positive and denoted by $A \gg 0$ if

$a_{ij} > 0$ for all i and j. Additionally, For a square matrix A, $\rho(A)$ denotes the spectral radius of A.

Some definitions, lemmas and theorems which will be used in the sequel are provided below.

Definition 1.1. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a *Z-matrix* if $a_{ij} \leq 0$ for $i \neq j$.

Definition 1.2. A Z-matrix A is called to be an *M-matrix* if A is nonsingular and $A^{-1} \geq 0$.

Definition 1.3. Let $A \in \mathbb{R}^{n\times n}$. The representation $A = M - N$ is called a *splitting* of A if M is nonsingular. The splitting $A = M - N$ is called
(a) convergent if $\rho(M^{-1}N) < 1$;
(b) an *M-splitting* of A if M is an M-matrix and $N \geq 0$.

Theorem 1.4. [13, Lemma 1.4] *Let A be a nonnegative matrix.*
(a) If $\alpha x \leq Ax$ for some nonnegative vector $x \neq 0$, then $\alpha \leq \rho(A)$.
(b) If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is positive.

Lemma 1.5. [13, Lemma 1.5] *Let $A = M - N$ be an M-splitting of A. Then, $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.*

Lemma 1.6. [13, Lemma 1.6] *Let A be a Z-matrix. Then, A is an M-matrix if and only if there is a positive vector x such that $Ax \gg 0$.*

Lemma 1.7. [12, Lemma 1.7] *Let $A \geq 0$. Then, $\alpha > \rho(A)$ if and only if $\alpha I - A$ is nonsingular and $(\alpha I - A)^{-1} \geq 0$.*

Lemma 1.8. [12, Lemma 2.2] *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an M-matrix. Then, there exists $\epsilon_0 > 0$ such that, for any $0 < \epsilon \leq \epsilon_0$, $A(\epsilon) = (a_{ij}(\epsilon))$ is also an M-matrix, where*

$$a_{ij}(\epsilon) = \begin{cases} a_{ij}, & a_{ij} \neq 0, \\ -\epsilon, & a_{ij} = 0. \end{cases}$$

Theorem 1.9. [1, Theorem 6.9] *Let A be an M-matrix that is partitioned in block matrix form $A = (A_{ij})$. Then*
(a) The matrices A_{ii} on the diagonal of A are M-matrices.
(b) The block lower and upper triangular parts of A are M-matrices.

In continuation, we review and give a lemma which is used in the paper. Consider the system of linear equations

$$\bar{A}\bar{x} = \bar{b}, \tag{1.3}$$

where $\bar{A} = (\bar{A}_{ij})$ is a nonsingular matrix. We assume that $\bar{A}$ has the splitting $\bar{A} = \bar{D} - \bar{L} - \bar{U}$ where $\bar{D}$ is the block diagonal matrix, $-\bar{L}$ and $-\bar{U}$ are strictly block lower and strictly block upper triangular matrices, respectively. With applying BAOR method for (1.3) the corresponding iteration matrix is denoted by

$$\bar{T}_{r,w} = (\bar{D} - r\bar{L})^{-1}[(1-w)\bar{D} + (w-r)\bar{L} + w\bar{U}].$$

system of linear equations (1.3) can also be presented by the form

$$\bar{A}\bar{D}^{-1}y = \bar{b}, \tag{1.4}$$

where $\bar{D}^{-1}y = \bar{x}$. Denote $\tilde{A} = \bar{A}D^{-1}$ has the usual splitting $\tilde{A} = I - \tilde{L} - \tilde{U}$ where I is identity matrix, $-\tilde{L}$ and $-\tilde{U}$ are strictly block lower and strictly block upper triangular matrices, respectively. With applying BAOR method for (1.4) the corresponding iteration matrix is denoted by

$$\tilde{T}_{r,w} = (I - r\tilde{L})^{-1}[(1-w)I + (w-r)\tilde{L} + w\tilde{U}].$$

The relation of spectral radii of iteration matrices $\bar{T}_{r,w}$ and $\tilde{T}_{r,w}$ is given by the next lemma.

Lemma 1.10. *Let A be a nonsingular Z-matrix blocked in the form $A = (A_{ij})$, Let also, $\bar{T}_{r,w}$ and $\tilde{T}_{r,w}$ be the iteration matrices of BAOR method given in* (1.3) *and* (1.4)*, respectively. If $\bar{D} = \operatorname{diag}(\bar{A}_{11}, \ldots, \bar{A}_{pp})$ is a nonsingular matrix, then $\rho(\tilde{T}_{r,w}) = \rho(\bar{T}_{r,w})$.*

Proof. Since $\bar{D} = \operatorname{diag}(\bar{A}_{11}, \ldots, \bar{A}_{pp})$ is a nonsingular matrix, then $\tilde{T}_{r,w}$ and $\bar{T}_{r,w}$ exist. Next,

$$\tilde{A} = I - \tilde{L} - \tilde{U} = (\bar{D} - \bar{L} - \bar{U})\bar{D}^{-1},$$

and

$$\begin{aligned} \tilde{T}_{r,w} &= (I - r\tilde{L})^{-1}[(1-w)I + (w-r)\tilde{L} + w\tilde{U}] \\ &= (I - r\bar{L}\bar{D}^{-1})^{-1}[(1-w)I + (w-r)\bar{L}\bar{D}^{-1} + w\bar{U}\bar{D}^{-1}] \\ &= \bar{D}(\bar{D} - r\bar{L})^{-1}[(1-w)\bar{D} + (w-r)\bar{L} + w\bar{U}]\bar{D}^{-1} \\ &= \bar{D}\ \bar{T}_{r,w}\ \bar{D}^{-1}, \end{aligned}$$

Therefore $\rho(\tilde{T}_{r,w}) = \rho(\bar{T}_{r,w})$. □

2. Preconditioned block AOR iterative method with $P^{(i)}$

Let $P^{(i)} = I + S^{(i)}$. If we apply the preconditioner $P^{(i)}$ to the system (1.1), then the coefficient matrix of the system can be written as

$$P^{(i)}A = (I + S^{(i)})A = I - L - U + S^{(i)} - S^{(i)}L - S^{(i)}U = D^{(i)} - L^{(i)} - U^{(i)},$$

where $D^{(i)}$, $L^{(i)}$ and $U^{(i)}$, are the block diagonal, block strictly lower triangular and block strictly upper triangular matrices, respectively. Denote $S^{(i)}L$ by $S^{(i)}L = S_1 + S_2 + S_3$, where S_1, S_2 and S_3 are the block diagonal, block strictly lower triangular and block strictly upper triangular matrices of $S^{(i)}L$, respectively. Then

$$D^{(i)} = I - S_1, \qquad L^{(i)} = L + S_2, \qquad U^{(i)} = U - S^{(i)} + S^{(i)}U + S_3. \tag{2.1}$$

Now, if $D^{(i)} - rL^{(i)}$ is nonsingular, then the iteration matrix of the BAOR iterative method with the preconditioner $P^{(i)}$ is of the form

$$T^{(i)}_{r,w} = (D^{(i)} - rL^{(i)})^{-1}[(1-w)D^{(i)} + (w-r)L^{(i)} + wU^{(i)}]. \tag{2.2}$$

Theorem 2.1. *Let $A = (A_{ij})$ be an M-matrix and $\alpha \in [0,1]$. Then $P^{(i)}A,\ i = 1, 2, \ldots, p-1$, is also an M-matrix.*

Proof. Let A be an M-matrix. Then

$$P^{(i)}A = \begin{pmatrix} C_{11} & \ldots & C_{1,p-i} & C_{1,p-i+1} & \ldots & C_{1p} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ C_{p-i,1} & \ldots & C_{p-i,p-i} & C_{p-i,p-i+1} & \ldots & C_{p-i,p} \\ A_{p-i+1,1} & \ldots & A_{p-i+1,p-i} & I_{p-i+1,p-i+1} & \ldots & A_{p-i+1,p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{p1} & \ldots & A_{p,n-i} & A_{p,n-i+1} & \ldots & I_{pp} \end{pmatrix},$$

where

$$C_{jk} = \begin{cases} A_{jk} - \alpha A_{j,j+i} A_{j+i,k}, & 1 \leq j \neq k \leq p-i, 1 \leq k \neq j+i \leq p, \\ I_{jj} - \alpha A_{j,j+i} A_{j+i,j}, & j = k, \\ (1-\alpha) A_{j,j+i}, & k = j+i. \end{cases}$$

Since A is a Z-matrix, for $i \neq j$ we have $A_{ij} \leq 0$. We also have $C_{ij} \leq 0,\ i \neq j$, and therefore, $P^{(i)}A$ is a Z-matrix. By Lemma 1.6 there is a positive vector x such that $Ax \gg 0$. On the other hand, $P^{(i)}Ax \gg 0$. Again, by Lemma 1.6 we conclude that the matrix $P^{(i)}A$ is an M-matrix. □

Theorem 2.2. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonsingular Z-matrix, $0 \leq r \leq w \leq 1$, $w \neq 0$ and $\alpha \in [0, 1]$. For $i = 1, 2, \ldots, p-1$,*
(a) If $\rho(T_{r,w}) < 1$, then $\rho(T^{(i)}_{r,w}) \leq \rho(T_{r,w}) < 1$.
(b) If $\rho(T_{r,w}) > 1$ and $\rho(A_{j,j+i}A_{j+i,j}) < 1$ for $1 \leq j \leq p-i$, then $\rho(T^{(i)}_{r,w}) \geq \rho(T_{r,w}) > 1$.

Proof. Consider the following splittings

$$A = M - N, \qquad P^{(i)}A = E^{(i)} - F^{(i)},$$

with

$$\begin{aligned} M &= \frac{1}{w}(I - rL), \quad N = \frac{1}{w}[(1-w)I + (w-r)L + wU], \\ E^{(i)} &= \frac{1}{w}(D^{(i)} - rL^{(i)}), \quad F^{(i)} = \frac{1}{w}[(1-w)D^{(i)} + (w-r)L^{(i)} + wU^{(i)}]. \end{aligned}$$

Since A is a nonsingular Z-matrix, $w \neq 0$, $0 \leq r \leq w \leq 1$, by lemma 1.7, it is clear that $M = \frac{1}{w}(I - rL)$ is an M-matrix. Therefore $A = M - N$ is an M-splitting. We have

$$\begin{aligned} T_{r,w} &= (I - rL)^{-1}[(1-w)I + (w-r)L + wU] \\ &= [\ I + (rL) + (rL)^2 + \cdots\ ][(1-w)I + (w-r)L + wU] \\ &\geq (1-w)I + (w-r)L + wU + r(1-w)L \\ &= (1-w)I + w(1-r)L + wU \geq 0. \end{aligned} \tag{2.3}$$

By Theorem 1.5 in [12] there is a nonnegative vector x such that

$$T_{r,w}x = \rho(T_{r,w})x.$$

We denote $\rho(T_{r,w})$ by λ. From the expression of $T_{r,w}$, we obtain the following equality

$$[(1-w)I + (w-r)L + wU]x = \lambda(I - rL)x,$$

which is equivalent to

$$[(1 - w - \lambda)I + (w - r + \lambda r)L + wU]x = 0, \tag{2.4}$$

and

$$(\lambda - 1)(I - rL)x = w(L + U - I)x. \tag{2.5}$$

From (2.1), (2.4) and (2.5), we have

$$
\begin{aligned}
&T^{(i)}_{r,w}x - \lambda x \\
&\quad = (D^{(i)} - rL^{(i)})^{-1}[(1-w)D^{(i)} + (w-r)L^{(i)} + wU^{(i)} - \lambda(D^{(i)} - rL^{(i)})]x \\
&\quad = (D^{(i)} - rL^{(i)})^{-1}[(1-w-\lambda)D^{(i)} + (w-r+\lambda r)L^{(i)} + wU^{(i)}]x \\
&\quad = (D^{(i)} - rL^{(i)})^{-1}[(1-w-\lambda)(I-S_1) + (w-r+\lambda r)(L+S_2) \\
&\qquad + w(U - S^{(i)} + S^{(i)}U + S_3)]x \\
&\quad = (D^{(i)} - rL^{(i)})^{-1}\{[(1-w-\lambda)I + (w-r+\lambda r)L + wU] \\
&\qquad + [-(1-w-\lambda)S_1 + (w-r+\lambda r)S_2 + w(S^{(i)}U - S^{(i)} + S_3)\}x \\
&\quad = (D^{(i)} - rL^{(i)})^{-1}[(\lambda-1)S_1 + wS_1 + r(\lambda-1)S_2 + wS_2 + w(S^{(i)}U - S^{(i)} + S_3)]x \\
&\quad = (D^{(i)} - rL^{(i)})^{-1}[(\lambda-1)S_1 + r(\lambda-1)S_2 + w(S_1 + S_2 + S_3 + S^{(i)}U - S^{(i)})]x \\
&\quad = (D^{(i)} - rL^{(i)})^{-1}[(\lambda-1)S_1 + r(\lambda-1)S_2 + w(S^{(i)}L + S^{(i)}U - S^{(i)})]x \\
&\quad = (D^{(i)} - rL^{(i)})^{-1}[(\lambda-1)S_1 + r(\lambda-1)S_2 + wS^{(i)}(L + U - I)]x \\
&\quad = (D^{(i)} - rL^{(i)})^{-1}[(\lambda-1)S_1 + r(\lambda-1)S_2 + (\lambda-1)S^{(i)}(I - rL)]x \\
&\quad = (D^{(i)} - rL^{(i)})^{-1}[(\lambda-1)S_1 + r(\lambda-1)S_2 + (\lambda-1)S^{(i)} - r(\lambda-1)S^{(i)}L]x \\
&\quad = (D^{(i)} - rL^{(i)})^{-1}[(\lambda-1)S_1 + r(\lambda-1)(S_2 - S^{(i)}L) + (\lambda-1)S^{(i)}]x \\
&\quad = (D^{(i)} - rL^{(i)})^{-1}[(\lambda-1)(1-r)S_1 + (\lambda-1)S^{(i)}]x \\
&\quad = (\lambda-1)(D^{(i)} - rL^{(i)})^{-1}[(1-r)S_1 + S^{(i)}]x. \qquad (2.6)
\end{aligned}
$$

Proof of (a): If $\lambda < 1$, then by Lemma 1.5, A is an M-matrix. Hence by Theorem 2.1 $P^{(i)}A$ is an M-matrix and by Theorem 1.9 $D^{(i)}$ is also an M-matrix. Since $(rD^{(i)})^{-1}L^{(i)} \geq 0$ is a strictly lower triangular matrix and $\rho(r(D^{(i)})^{-1}L^{(i)}) = 0 < 1$, by lemma 1.7, we have $(\ I - r(D^{(i)})^{-1}L^{(i)}\ )^{-1} \geq 0$. Then

$$(D^{(i)} - rL^{(i)})^{-1} = (\ I - r(D^{(i)})^{-1}L^{(i)}\ )^{-1}(D^{(i)}\ )^{-1} \geq 0,$$

which means that $E^{(i)}$ is an M-matrix. Furthermore, $F^{(i)} \geq 0$. This shows that $P^{(i)}A = E^{(i)} - F^{(i)}$ is also an M-splitting. Therefore, by Lemma 1.5, we see that $\rho(T^{(i)}_{r,w}) = \rho((E^{(i)})^{-1}F^{(i)}) < 1$. We first consider the case when A is irreducible. When $0 \leq r < 1$, from (2.3), we can see that $T_{r,w}$ is also irreducible. Hence, by Lemma 1.6 in [12] we have $\lambda > 0$ and the vector x is positive. Now, from Eq. (2.6) we conclude that

$$T^{(i)}_{r,w}x \leq \lambda x,$$

and by using Theorem 1.4 we get

$$\rho(T^{(i)}_{r,w}) \leq \lambda = \rho(T_{r,w}).$$

If $r = 1$, then $w = r = 1$ and

$$\rho(T^{(i)}_{1,1}) = \lim_{r\to 1^-} \rho(T^{(i)}_{r,1}) \leq \lim_{r\to 1^-} \rho(T_{r,1}) = \rho(T_{1,1}) < 1.$$

Now, we consider the case when A is reducible. According to Lemma 1.8, for any sufficiently small positive number $\epsilon > 0$, $A(\epsilon)$ is an M-matrix and irreducible. By the proof presented above, we have

$$\rho(T^{(i)}_{r,w}) = \lim_{\epsilon\to 0^+} \rho(T^{(i)}_{r,w}(\epsilon)) \leq \lim_{\epsilon\to 0^+} \rho(T_{r,w}(\epsilon)) = \rho(T_{r,w}) < 1.$$

This completes the first part of the theorem.

Proof of (b): If $\lambda > 1$, then by the assumption $\rho(A_{j,j+i}A_{j+i,j}) < 1,\ 1 \leq j \leq p-i,\ i = 1,2,\ldots,p-1$, and by the Lemma 1.7, the blocks C_{jj} are M-matrices, and hence

$$(D^{(i)})^{-1} = diag((C_{11})^{-1},\ldots,(C_{p-i,p-i})^{-1}, I_{p-i+1,p-i+1},\ldots,I_{pp}) \geq 0.$$

Therefore

$$(D^{(i)} - rL^{(i)})^{-1} = (I - r(D^{(i)})^{-1}L^{(i)})^{-1}\ (D^{(i)})^{-1} \geq 0.$$

Now, from Eq. (2.6) we conclude that

$$T^{(i)}_{r,w}x \geq \lambda x,$$

and by using Theorem 1.4 we get

$$\rho(T^{(i)}_{r,w}) \geq \rho(T_{r,w}) > 1.$$

This proves the second part of the theorem. □

3. Preconditioned block AOR iterative method with $P^{(-i)}$

Let $P^{(-i)} = I + S^{(-i)}$. Applying $P^{(-i)}$ to (1.1), $P^{(-i)}A$ can be written as

$$P^{(-i)}A = (I + S^{(-i)})A = I - L - U + S^{(-i)} - S^{(-i)}L - S^{(-i)}U = D^{(-i)} - L^{(-i)} - U^{(-i)},$$

where $D^{(-i)}$, $L^{(-i)}$ and $U^{(-i)}$, are the block diagonal, block strictly lower triangular and block strictly upper triangular matrices, respectively. Denote $S^{(-i)}U$ by $S^{(-i)}U = S'_1 + S'_2 + S'_3$, where S'_1, S'_2 and S'_3 are the block diagonal, block strictly lower triangular and strictly upper triangular matrices of $S^{(-i)}U$, respectively. Then

$$D^{(-i)} = I - S'_1, \qquad L^{(-i)} = L - S^{(-i)} + S^{(-i)}L + S'_2, \qquad U^{(-i)} = U + S'_3. \tag{3.1}$$

Now, if $D^{(-i)} - rL^{(-i)}$ is nonsingular, then the iteration matrix of the BAOR iterative method with the preconditioner $P^{(-i)}$ is of the form

$$T^{(-i)}_{r,w} = (D^{(-i)} - rL^{(-i)})^{-1}[(1-w)D^{(-i)} + (w-r)L^{(-i)} + wU^{(-i)}].$$

Theorem 3.1. *Let $A = (A_{ij})$ be an M-matrix and $\alpha \in [0,1]$. Then $P^{(-i)}A,\ i = 1,2,\ldots,p-1$, is also an M-matrix.*

Proof. Let A be an M-matrix. Then

$$P^{(-i)}A = \begin{pmatrix} I_{11} & \ldots & A_{1,i} & A_{1,i+1} & \ldots & A_{1p} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ A_{i,1} & \ldots & I_{i,i} & A_{i,i+1} & \ldots & A_{i,p} \\ C_{i+1,1} & \ldots & C_{i+1,i} & C_{i+1,i+1} & \ldots & C_{i+1,p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{p1} & \ldots & C_{p,i} & C_{p,i+1} & \ldots & C_{pp} \end{pmatrix},$$

where,

$$C_{jk} = \begin{cases} A_{jk} - \alpha A_{j,j-i}A_{j-i,k}, & i+1 \leq j \neq k \leq p, 1 \leq k \neq j-i \leq p, \\ I_{jj} - \alpha A_{j,j-i}A_{j-i,j}, & j = k, \\ (1-\alpha)A_{j,j-i}, & k = j-i. \end{cases}$$

Similar to Theorem 2.1 the proof can be completed. □

Theorem 3.2. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonsingular Z-matrix, $0 \le r \le w \le 1$, $w \ne 0$ and $\alpha \in [0,1]$. For $i = 1, 2, \ldots, p-1$*
(a) If $\rho(T_{r,w}) < 1$, then $\rho(T_{r,w}^{(-i)}) \le \rho(T_{r,w}) < 1$.
(b) If $\rho(T_{r,w}) > 1$, $\rho(A_{j,j-i}A_{j-i,j}) < 1$ for $i+1 \le j \le p$ then $\rho(T_{r,w}^{(-i)}) \ge \rho(T_{r,w}) > 1$.

Proof. Assume that

$$\begin{aligned} M &= \frac{1}{w}(I - rL), \qquad N = \frac{1}{w}[(1-w)I + (w-r)L + wU], \\ E^{(-i)} &= \frac{1}{w}(D^{(-i)} - rL^{(-i)}), \qquad F^{(-i)} = \frac{1}{w}[(1-w)D^{(-i)} + (w-r)L^{(-i)} + wU^{(-i)}]. \end{aligned}$$

Then

$$A = M - N, \qquad P^{(-i)}A = E^{(-i)} - F^{(-i)}.$$

It is easy to see that under the assumptions of the theorem, $A = M - N$ is an M-splitting. Similar to Theorem 2.2 there is a nonnegative vector x such that

$$T_{r,w}x = \rho(T_{r,w})x.$$

We denote $\rho(T_{r,w})$ by λ. From the expression of $T_{r,w}$, we obtain the following equality

$$[(1-w)I + (w-r)L + wU]x = \lambda(I - rL)x$$

which is equivalent to

$$[(1-w-\lambda)I + (w-r+\lambda r)L + wU]x = 0, \tag{3.2}$$

and

$$(\lambda - 1)(I - rL)x = w(L + U - I)x. \tag{3.3}$$

From (3.1), (3.2) and (3.3), we have

$$\begin{aligned} &T_{r,w}^{(-i)}x - \lambda x \\ &\quad= [(1-w)D^{(-i)} + (w-r)L^{(-i)} + wU^{(-i)} - \lambda(D^{(-i)} - rL^{(-i)})]x \\ &\quad= (D^{(-i)} - rL^{(-i)})^{-1}[(1-w-\lambda)D^{(-i)} + (w-r+\lambda r)L^{(-i)} + wU^{(-i)}]x \\ &\quad= (D^{(-i)} - rL^{(-i)})^{-1}[(1-w-\lambda)(I - S_1') + (w-r+\lambda r)(L - S^{(-i)} + S^{(-i)}L + S_2') \\ &\qquad +w(U + S_3')]x \\ &\quad= (D^{(-i)} - rL^{(-i)})^{-1}\{[(1-w-\lambda)I + (w-r+\lambda r)L + wU] \\ &\qquad +[-(1-w-\lambda)S_1' + (w-r+\lambda r)(-S^{(-i)} + S^{(-i)}L + S_2') + wS_3']\}x \\ &\quad= (D^{(-i)} - rL^{(-i)})^{-1}[(\lambda-1)S_1' + r(\lambda-1)(-S^{(-i)} + S^{(-i)}L + S_2') \\ &\qquad +w(-S^{(-i)} + S^{(-i)}L) + w(S_1' + S_2' + S_3')]x \\ &\quad= (D^{(-i)} - rL^{(-i)})^{-1}[(\lambda-1)S_1' + r(\lambda-1)(-S^{(-i)} + S^{(-i)}L + S_2') \\ &\qquad +wS^{(-i)}(L + U - I)]x \\ &\quad= (D^{(-i)} - rL^{(-i)})^{-1}[(\lambda-1)S_1' + r(\lambda-1)(-S^{(-i)} + S^{(-i)}L + S_2') \\ &\qquad +(\lambda-1)S^{(-i)}(I - rL)]x \\ &\quad= (\lambda-1)(D^{(-i)} - rL^{(-i)})^{-1}[S_1' + r(-S^{(-i)} + S^{(-i)}L + S_2') + S^{(-i)}(I - rL)]x \\ &\quad= (\lambda-1)(D^{(-i)} - rL^{(-i)})^{-1}[S_1' + rS_2' + (1-r)S^{(-i)}]x. \end{aligned}$$

Similar to Theorem 2.2 the proof can be completed. □

4. Multi-level Preconditioned block AOR iterative method with $P^{(i)}$ and $P^{(-i)}$

In this section, we propose a multi-level preconditioner and a comparison theorem concerning the proposed preconditioner.

Let $A^{(0)} = A$. For $i = 1, 2, \dots, p-1$ by defining a matrix of the form

$$Q^{(i)} = \begin{pmatrix} (I_{11} - \alpha A_{1,1+i}A^{(-(i-1))}_{1+i,1})^{-1} & & & & & \\ & \ddots & & & & \\ & & (I_{p-i,p-i} - \alpha A_{p-i,p}A^{(-(i-1))}_{p,p-i})^{-1} & & & \\ & & & I_{p-i+1,p-i+1} & & \\ & & & & \ddots & \\ & & & & & I_{p,p} \end{pmatrix},$$

the matrix $P^{(i)}A^{(-(i-1))}Q^{(i)}$ has the form

$$A^{(i)} = P^{(i)}A^{(-(i-1))}Q^{(i)} = \begin{pmatrix} I_{11} & A^{(i)}_{12} & \dots & A^{(i)}_{1p} \\ A^{(i)}_{21} & I_{22} & \dots & A^{(i)}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A^{(i)}_{p1} & A^{(i)}_{p2} & \dots & I_{pp} \end{pmatrix},$$

and also by defining a matrix of the form

$$Q^{(-i)} = \begin{pmatrix} I_{11} & & & & & \\ & \ddots & & & & \\ & & I_{i,i} & & & \\ & & & (I_{i+1,i+1} - \alpha A_{i+1,1}A^{(i)}_{1,i+1})^{-1} & & \\ & & & & \ddots & \\ & & & & & (I_{pp} - \alpha A_{p,p-i}A^{(i)}_{p-i,p})^{-1} \end{pmatrix},$$

the matrix $P^{(-i)}A^{(i)}Q^{(-i)}$ has the form

$$A^{(-i)} = P^{(-i)}A^{(i)}Q^{(-i)} = \begin{pmatrix} I_{11} & A^{(-i)}_{12} & \dots & A^{(-i)}_{1p} \\ A^{(-i)}_{21} & I_{22} & \dots & A^{(-i)}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A^{(-i)}_{p1} & A^{(-i)}_{p2} & \dots & I_{pp} \end{pmatrix}.$$

Two preconditioners $P^{(i)}$ and $P^{(-i)}$ can be applied step-step for all diagonals of A. For the linear system (1.1) we consider its preconditioned form

$$P^{(l)}_m Ax = P^{(l)}_m b, \tag{4.1}$$

with the preconditioner

$$P_m^{(l)} = \begin{cases} P^{(l)}P^{(-(l-1))}P^{(l-1)}\cdots P^{(-1)}P^{(1)}, & l>0 \\ P^{(l)}P^{(-l)}P^{(l+1)}P^{(-(l+1))}\cdots P^{(-1)}P^{(1)}, & l<0 \end{cases}$$

that $l=-(p-1),\ldots,-2,-1,1,2,\ldots,p-1$.

The multi-level preconditioned block AOR method of (1.1), i.e., the block AOR method of (4.1), is defined as

$$x^{(k+1)} = \tilde{T}_{r,w}^{(l)}\, x^{(k)} + w(\tilde{D}^{(l)} - r\tilde{L}^{(l)})^{-1}b, \quad k=0,1,2,\ldots,$$

where

$$\tilde{T}_{r,w}^{(l)} = (\tilde{D}^{(l)} - r\tilde{L}^{(l)})^{-1}[(1-w)\tilde{D}^{(l)} + (w-r)\tilde{L}^{(l)} + w\tilde{U}^{(l)}],$$

is the iteration matrix of multi-level preconditioned block AOR system where $\tilde{D}^{(l)}$, $\tilde{L}^{(l)}$ and $\tilde{U}^{(l)}$, are the block diagonal, block strictly lower triangular and block strictly upper triangular matrices of $P_m^{(l)}A$, respectively.

We assume $A^{(0)}=A,\ y^{(0)}=x.$ Also we let $A^{(i)} = P^{(i)}A^{(-(i-1))}Q^{(i)}, A^{(-i)} = P^{(-i)}A^{(i)}Q^{(-i)}$, $i=1,2,\ldots,p-1$. Then the system (4.1) is transformed into the following equivalent form

$$\begin{cases} P^{(i)}A^{(-(i-1))}y^{(-(i-1))} = P_m^{(l)}b, \\ y^{(-(i-1))} = Q^{(i)}y^{(i)}, \\ P^{(-i)}A^{(i)}y^{(i)} = P_m^{(-l)}b, \\ y^{(i)} = Q^{(-i)}y^{(-i)}, \end{cases}$$

for $l=i=1,2,\ldots,p-1$. We apply the BAOR method to the following system

$$A^{(-i)}y^{(-i)} = P_m^{(-l)}b, \qquad for\ \ l=i=p-1,$$

and then

$$y^{(i)} = Q^{(-i)}y^{(-i)}, \quad y^{(-(i-1))} = Q^{(i)}y^{(i)}, \qquad for\ \ i=p-1,p-2,\ldots,1.$$

Theorem 4.1. *Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be a nonsingular Z-matrix, $0\le r\le w\le 1$, $w\neq 0$ and $\alpha\in[0,1]$.*
(a) If $\rho(T_{r,w})<1$, then

$$\rho(\tilde{T}_{r,w}^{(-(p-1))}) \le \rho(\tilde{T}_{r,w}^{(p-1)}) \le \cdots \le \rho(\tilde{T}_{r,w}^{(-2)} \le \rho(\tilde{T}_{r,w}^{(2)}) \le \rho(\tilde{T}_{r,w}^{(-1)} \le \rho(\tilde{T}_{r,w}^{(1)}) \le \rho(T_{r,w}) < 1,$$

(b) If $\rho(T_{r,w})>1, for\ i=1,2,\ldots,p-1, \rho(A_{j,j+i}A_{j+i,j}^{(-(i-1))})<1,\ 1\le j\le p-i,$ and $\rho(A_{j,j-i}A_{j-i,j}^{(i)})<1,\ i+1\le j\le p$, then

$$\rho(\tilde{T}_{r,w}^{(-(p-1))}) \ge \rho(\tilde{T}_{r,w}^{(p-1)}) \ge \cdots \ge \rho(\tilde{T}_{r,w}^{(-2)} \ge \rho(\tilde{T}_{r,w}^{(2)}) \ge \rho(\tilde{T}_{r,w}^{(-1)} \ge \rho(\tilde{T}_{r,w}^{(1)}) \ge \rho(T_{r,w}) > 1.$$

Proof. by Lemma 1.10, Theorem 2.2 and Theorem 3.2, we can prove the theorem. □

Table 1. Spectral radii of the block AOR method and multi-level preconditioned block AOR method

w	r	$l=0$	$l=1$	$l=-1$	$l=2$	$l=-2$
0.6	0.8	0.610764	0.608017	0.607443	0.604742	0.604164
0.8	0.6	0.519736	0.516680	0.515424	0.512405	0.511148
0.8	1.0	0.423800	0.418796	0.418778	0.413872	0.413847
1.0	0.8	0.351274	0.346695	0.345739	0.341238	0.340273
1.0	1.0	0.279750	0.273495	0.273473	0.267340	0.267309

Table 2. Spectral radii of the block AOR method and multi-level preconditioned block AOR method

w	r	$l=0$	$l=-5$	$l=-10$	$l=-15$	$l=-20$
0.6	0.8	0.610764	0.594597	0.579637	0.566005	0.553742
0.8	0.6	0.519736	0.498567	0.478537	0.459848	0.442633
0.8	1.0	0.423800	0.399647	0.378144	0.359511	0.343719
1.0	0.8	0.351274	0.324328	0.299395	0.276675	0.256237
1.0	1.0	0.279750	0.249559	0.222680	0.199389	0.179648

5. Numerical example

We consider the $n \times n$ coefficient matrix

$$A = \begin{pmatrix} 1 & q & r & s & q & \dots \\ s & 1 & q & r & \ddots & q \\ r & s & \ddots & \ddots & \ddots & s \\ q & \ddots & \ddots & 1 & q & r \\ s & \ddots & r & s & 1 & q \\ \dots & s & q & r & s & 1 \end{pmatrix},$$

where $q = -5/10n$, $r = -5/(10n+1)$, $s = -5/(10n+2)$ and $n = 300$. Obviously, A is an M-matrix. In continuation, we partition A into block matrix form $A = (A_{ij})$, where blocks A_{ij} have size 4×4. In our implementations we choose $\alpha = 0.8$.

In Tables 1 and 2 we denote the spectral radii of iteration matrices of basic iterative AOR method by $l = 0$. Also the spectral radii of iteration matrices for multi-level block preconditioned AOR method are listed when the parameter l is varying.

From Tables 1 and 2, it is seen that the spectral radii of iteration matrices are decreasing when l increases, which shows that the converge speeds of block preconditioned AOR methods are improved step-by-step by the multi-level block preconditioner.

6. Conclusion

We have presented a sequence of preconditioners for block accelerated overrelaxation (BAOR) iterative method. Some comparison theorems have been provided to show the convergence of the multi-level preconditioned block AOR method for solving a linear system whose coefficient matrix is a nonsingular Z-matrix. Numerical experiments show that the multi-level block preconditioners are efficient.

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