Two-stage waveform relaxation method for the initial value problems with non-constant coefficients

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Abstract

In this paper, we present a two-stage waveform relaxation method applied to the initial value problems for the linear systems of ordinary differential equations in the form $y'(t)+A(t)y(t) = f(t)$. By making use of the forward Euler method, we derive sufficient conditions for the convergence of this method, when $A(t)$ is M-matrix for every $t \in [t_0, T]$. Finally some numerical experiments are given to illustrate some of the theoretical results.

Key words: Two-stage, Waveform relaxation method, Euler method, Ordinary differential equations, Inner/outer, M-splitting.
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1. Introduction

Many scientific and engineering problems can be represented by linear systems of ordinary differential equations (ODEs) and differential-algebraic equations (DAEs). For instance, electrical networks, constrained mechanical systems of rigid bodies, control theory, singular perturbation and discretization of partial differential equations, etc. (see [4, 5, 6]). Several iterative methods have been investigated to solve this kind of problems. The two-stage iterative method was first proposed for solving systems of linear equations by Nichols [9] in 1973. After that, waveform relaxation (WR) methods have been developed in order to numerically solve systems of ODEs, hence two-stage waveform relaxation (TSWR) method has been studied for ODEs and DAEs (see [7, 2, 14, 13]).

In recent years, for an initial value problem (IVP) of ODEs in the form

$$\begin{cases} y'(t) + Ay(t) = f(t), \\ y(t_0) = y_0, \quad t \in [t_0, T], \end{cases}$$

where $A \in \mathbb{C}^{m\times m}$ is a nonsingular matrix and $f(t) : [t_0, T] \longrightarrow \mathbb{C}^m$ is supposed continuous, the TSWR method has been proposed and investigated. For instance,

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the convergence analysis for A being an M-matrix is given in [7]. In [13] the method is investigated when A is an H-matrix, and in [14] the convergence analysis is restricted to Hermitian positive definite matrices.

The main aim of this work is to extend the analysis of WR and TSWR methods to IVPs of systems of linear ODEs with non constant coefficients in the form

$$\begin{cases} y'(t) + A(t)y(t) = f(t), \\ y(t_0) = y_0, \quad t \in [t_0, T], \end{cases} \tag{1}$$

where $A(t) : [t_0, T] \longrightarrow \mathbb{R}^{m\times m}$ is a nonsingular M-matrix for every $t \in [t_0, T]$ with continuous entries and $f(t) : [t_0, T] \longrightarrow \mathbb{R}^m$ is supposed to be continuous. Similar to [7], we consider an M-splitting of the form $A(t) = C(t) - D(t)$, for the matrix $A(t)$ on $[t_0, T]$. In this case the WR continuous-time iteration can be given by

$$\begin{cases} y'^{k+1}(t) + C(t)y^{k+1}(t) = D(t)y^k(t) + f(t), \\ y^{k+1}(t_0) = y_0, \quad k = 0, 1, \dots. \end{cases} \tag{2}$$

Iterative method (2) is called the outer iteration. By using the splitting $C(t) = M(t) - N(t)$ an inner iteration is then generated where whole splitting $A(t) = M(t) - N(t) - D(t)$ is a composite M-splitting for the matrix $A(t)$ on $[t_0, T]$. In this way the continuous-time TSWR iterations are generated. For discretizing the continuous-time TSWR method we use the Euler method.

The rest of the paper is organized as follows. After achieving the discretized TSWR iteration in section 2, we recall some basic theories of iterative methods for linear algebraic systems in section 3. We investigate covergence analysis of the discretized TSWR method in section 4. Finally, some numerical experiments are given in section 5.

2. Two-stage waveform relaxation method

For later use we first give a brief description of the WR method. As we mentioned in the previous section, to solve IVP (1), the splitting $A(t) = C(t) - D(t)$ is put. Based on this splitting WR continuous-time iterations are defined in the form (2), such that $y^0(t)$ is a starting solution which is assumed here $y^0(t) \equiv y_0$, $t \in [t_0, T]$. For the given step size h and an equally spaced grid $\{t_0, t_1, \dots, t_N\}$ for $t \in [t_0, T]$, by applying the forward Euler method

$$y'(t) = \frac{y(t_{n+1}) - y(t_n)}{h}, \quad n = 0, 1, \dots, N-1,$$

discretized WR iterations can be written in the form

$$\begin{cases} (I + hC_{n+1})y^{k+1}_{n+1} = y^{k+1}_n + hD_{n+1}y^k_{n+1} + hf(t_{n+1}), \\ y^{k+1}_0 = y_0, \quad k = 0, 1, \dots, \quad n = 0, 1, \dots, N-1, \end{cases} \tag{3}$$

where y^k_n is approximation for $y^k(t_n)$ and for brevity of notation, $C(t_n)$ and $D(t_n)$ are denoted by C_n and D_n, respectively. By putting $k_{n+1} = I + hC_{n+1}$, we can formulate the discretized WR iterations as following.

Discretized waveform relaxation method:
for $k = 0, 1, \ldots, \quad n = 0, 1, \ldots, N$
$y_0^k = y_0, \quad y_n^0 = y_0$
for $n = 0, 1, \ldots, N-1$
$y_{n+1}^{k+1} = k_{n+1}^{-1}(y_n^{k+1} + hD_{n+1}y_{n+1}^k + hf(t_{n+1}))$
end for
end for

By substituting the splitting $C(t) = M(t) - N(t)$ in Eq. (2), the inner iterations are generated. Hence the TSWR continuous-time iterations are defined as

$$\begin{cases} z'^{v+1}(t) + M(t)z^{v+1}(t) = N(t)z^v(t) + D(t)y^k(t) + f(t), \\ z^{v+1}(t_0) = y^k(t_0) = y_0, \quad k = 0, 1, \ldots, \quad v = 0, 1, \ldots, v_k - 1, \end{cases} \tag{4}$$

where $z^0(t) = y^k(t)$ and $y^{k+1}(t) = z^{v_k}(t)$ for $t \in [t_0, T]$, so that v_k is the number of inner iterations for kth outer iteration. This procedure is called two-stage iterative method.

Similar to [7, 2, 14], we assume that the number of inner iterations steps is fixed for all outer iterations, for example $v_k \equiv s$, $k = 0, 1, \ldots$, where s is a positive integer. In this case, the TSWR method (4) is named as stationary. In Eq. (4), if $N(t) = 0$, then the TSWR iterative method leads to the WR iterative method.

For discretization, we use the forward Euler method for the given step size h and an equally spaced grid $\{t_0, t_1, \ldots, t_N\}$ on $[t_0, T]$. Hence the following iterations are generated

$$\begin{cases} (I + hM_{n+1})z_{n+1}^{v+1} = z_n^{v+1} + hN_{n+1}z_{n+1}^v + hD_{n+1}y_{n+1}^k + hf(t_{n+1}), \\ z_0^{v+1} = y_0^k = y_0, \quad k = 0, 1, \ldots, \quad v = 0, 1, \ldots, s-1, \end{cases} \tag{5}$$

where z_n^v, M_n and N_n are approximations for $z^v(t_n)$, $M(t_n)$ and $N(t_n)$ respectively. By putting $H_n = (I + hM_n)^{-1}$ and

$$b_n(v, k) = (I + hM_n)^{-1}(N_n z_n^v + D_n y_n^k + f(t_n)),$$

the discretized TSWR iteration is formulated in the following form.

Discretized two-stage waveform relaxation method:
for $k = 0, 1, \ldots, \quad n = 0, 1, \ldots, N$
$z_n^0 = y_n^k$
for $v = 0, 1, \ldots, s-1$
$z_0^{v+1} = y_0^k = y_0$
for $n = 0, 1, \ldots, N-1$
$z_{n+1}^{v+1} = H_{n+1}z_n^{v+1} + hb_{n+1}(v, k)$
end for
end for
$y_n^{k+1} = z_n^s, \quad n = 0, 1, \ldots N$
end for

Lemma 2.1. *Given an $m \times m$ real matrix H_n and a sequence $\{a_n\}_{n\in\mathbb{N}}$ of vectors in $\mathbb{R}^m$, for any $h \geq 0$ the solution of the difference equation*

$$a_n = H_n a_{n-1} + h b_n(v,k), \quad v = 0,\dots,s,\ k = 0,1,\dots,$$

can be expressed in the form

$$a_n = \prod_{j=1}^{n} H_j a_0 + h \sum_{j=1}^{n-1} (\prod_{i=j+1}^{n} H_i) b_j(v,k) + h b_n(v,k),$$

for $v = 0,\dots,s,\ k = 0,1,\dots$.

Proof. The proof follows by writing

$$\begin{aligned} a_n &= H_n a_{n-1} + h b_n(v,k) = H_n H_{n-1} a_{n-2} + h H_n b_{n-1}(v,k) + h b_n(v,k) \\ &\vdots \\ &= \prod_{j=1}^{n} H_j a_0 + h[\prod_{i=2}^{n} H_i b_1(v,k) + \cdots + \prod_{i=n}^{n} H_i b_{n-1}(v,k)] + h b_n(v,k) \\ &= \prod_{j=1}^{n} H_j a_0 + h \sum_{j=1}^{n-1} (\prod_{i=j+1}^{n} H_i) b_j(v,k) + h b_n(v,k). \qquad \square \end{aligned}$$

By applying Lemma 2.1 for

$$z_{n+1}^{v+1} = H_{n+1} z_n^{v+1} + h b_{n+1}(v,k) \tag{6}$$

the sequence $\{z_n^{v+1}\}_{n=1,\dots,N}$ can be written in the form

$$z_n^{v+1} = \prod_{j=1}^{n} H_j z_0^{v+1} + h \sum_{j=1}^{n-1} (\prod_{i=j+1}^{n} H_i) b_j(v,k) + h b_n(v,k).$$

By substituting $z_0^{v+1} = y_0$ and $b_j(v,k)$ by its expression, we can obtain the following recurrence relation

$$z_n^{v+1} = G_n z_n^v + L_n y_n^k + h \sum_{j=1}^{n-1} (\prod_{i=j+1}^{n} H_i) H_j (N_j z_j^v + D_j y_j^k) + g_n, \tag{7}$$

where the matrices G_n and L_n are given by

$$G_n = h(I + M_n)^{-1} N_n, \quad L_n = h(I + hM_n)^{-1} D_n, \tag{8}$$

and

$$g_n = h \sum_{j=1}^{n-1} (\prod_{i=j+1}^{n} H_i) H_j f(t_j) + \prod_{j=1}^{n} H_j y_0 + h H_n f(t_n).$$

Now, by setting $\overline{z}_n^v = [z_1^{v^T}, z_2^{v^T}, \ldots, z_n^{v^T}]^T$, $\overline{y}_n^k = [y_1^{k^T}, y_2^{k^T}, \ldots, y_n^{k^T}]^T$, $\overline{g}_n = [g_1^T, g_2^T, \ldots, g_n^T]^T$,

$$U_n = \begin{pmatrix} G_n & & & \\ h(\prod_{i=n}^{n} H_i)H_{n-1}N_{n-1} & G_n & & \\ h(\prod_{i=n-1}^{n} H_i)H_{n-2}N_{n-2} & h(\prod_{i=n}^{n} H_i)H_{n-1}N_{n-1} & G_n & \\ \vdots & \vdots & & \ddots \\ h(\prod_{i=2}^{n} H_i)H_1N_1 & h(\prod_{i=3}^{n} H_i)H_2N_2 & \cdots & G_n \end{pmatrix},$$

and

$$P_n = \begin{pmatrix} L_n & & & \\ h(\prod_{i=n}^{n} H_i)H_{n-1}D_{n-1} & L_n & & \\ h(\prod_{i=n-1}^{n} H_i)H_{n-2}D_{n-2} & h(\prod_{i=n}^{n} H_i)H_{n-1}D_{n-1} & L_n & \\ \vdots & \vdots & & \ddots \\ h(\prod_{i=2}^{n} H_i)H_1D_1 & h(\prod_{i=3}^{n} H_i)H_2D_2 & \cdots & L_n \end{pmatrix},$$

we can write Eq. (7) in the following form

$$\overline{z}_n^{v+1} = U_n\overline{z}_n^v + P_n\overline{y}_n^k + \overline{g}_n. \tag{9}$$

For a fixed number s of inner iterations the solution of Eq. (9) can be written as

$$\overline{z}_n^s = U_n^s\overline{z}_n^0 + \sum_{v=0}^{s-1} U_n^v P_n\overline{y}_n^k + \sum_{v=0}^{s-1} U_n^v\overline{g}_n.$$

In order to obtain a general expression for approximations z_n^s, we can extract the last component of $\overline{z}_n^s$ as

$$\begin{aligned} z_n^s &= G_n^s z_n^0 + \sum_{j=1}^{n-1} U_{n,j}^s z_j^0 + \sum_{v=0}^{s-1}[G_n^v L_n y_n^k + \sum_{j=1}^{n-1}\sum_{i=1}^{n} U_{n,i}^v P_{i,j} y_j^k] \\ &\quad + \sum_{v=0}^{s-1}[G_n^v g_n + \sum_{j=1}^{n-1} U_{n,j}^v g_j], \end{aligned}$$

where with $U_{n,j}^v$ and $P_{n,j}$ we indicated the (n, j)-blocks of U_n^v and P_n, respectively. By putting

$$T_s = G_n^s + \sum_{v=0}^{s-1} G_n^v L_n, \quad S_s = \sum_{v=0}^{s-1} G_n^v, \tag{10}$$

and

$$p_{s,n}(k)=\sum_{j=1}^{n-1}U_{n,j}^{v}y_j^k+\sum_{v=0}^{s-1}\sum_{j=1}^{n-1}\sum_{i=1}^{n}U_{n,i}^{v}P_{i,j}y_j^k+\sum_{v=0}^{s-1}\sum_{j=1}^{n-1}U_{n,j}^{v}g_j,$$

and substituting $z_n^0=y_n^k$, $z_n^s=y_n^{k+1}$ we can write a general recurrence relation on index k, for computation of the sequence of the approximations $\{y_n^k\}_{k=0,1,\ldots}$ at the same grid point t_n, as

$$y_n^{k+1}=T_s y_n^k+S_s g_n+p_{s,n}(k). \tag{11}$$

Representation of the TSWR method in the form (11), which is equivalent with Eq. (6), is useful, because it makes easier to investigate convergence analysis in each grid point t_n, based on the study of convergence properties of the matrix T_s.

3. Definitions and lemmas

In order to investigate convergence of the TSWR method, in this section we recall some definitions and basic concepts from the stationary iterations.

An $m\times m$ real matrix $A=(a_{ij})$ is said to be nonnegative, denoted by $A\geq 0$, if its all entries are nonnegative. A is called inverse nonnegative, if its inverse exists and $A^{-1}\geq 0$. A is called an M-matrix if it can be written in the form $A=sI-B$, where $B\geq 0$, $s>0$ and $\rho(B)\leq s$. It can be seen that the matrix $A=(a_{ij})$ is a nonsingular M-matrix if and only if $a_{ij}\leq 0$, for $i\neq j$ and $a_{ii}>0$, $i=1,\ldots,m$ and A is inverse nonnegative.

The matrix function $A(t)$ for $t\in[t_0,T]$ is said to be an M-matrix if it is an M-matrix for every $t\in[t_0,T]$. In other words, the matrix function $A(t)$ is an M-matrix if it can be written in the form $A(t)=s(t)I-B(t)$, so that $B(t)\geq 0$, $s(t)>0$ and $\rho(B(t))\leq s(t)$ for every $t\in[t_0,T]$.

Definition 3.1. *For nonsingular matrices A and C, we say that $A=C-D$ is*

(a) *a weak regular splitting if $C^{-1}\geq 0$ and $C^{-1}D\geq 0$;*

(b) *a regular splitting if $C^{-1}\geq 0$ and $D\geq 0$;*

(c) *an M-splitting if C is an M-matrix and $D\geq 0$.*

Similar to this definition, $A(t)=C(t)-D(t)$ is said to be an M-splitting for the matrix function $A(t)$, $t\in[t_0,T]$ if it is an M-splitting for every $t\in[t_0,T]$. Similarly, other definitions can be extended to matrix functions $A(t)$. Obviously an M-splitting is also a regular splitting and weak regular splitting.

Definition 3.2. *Given a nonsingular matrix A, we say that $A=M-N-D$ is a composite M-splitting if $C=M-N$ and $A=C-D$ are both M-splitting.*

Lemma 3.3. *([3]) Let $A=C-D$ be a regular splitting of A. Then A is nonsingular and $A^{-1}\geq 0$ if and only if $\rho(C^{-1}D)<1$.*

Lemma 3.4. *([7]) Consider the iterative process*

$$y^{k+1} = Ty^k + Sg + p_k, \tag{12}$$

where S is nonsingular, $\rho(T) < 1$ and $\lim_{k\to\infty} p_k = p$. Then there exists a unique nonsingular matrix A and a unique splitting $A = C - D$ such that $C = S^{-1}$, $T = C^{-1}D$ and Eq. (12) converges to the solution of the system $Ay = b$, where $b = g + S^{-1}p$.

4. Convergence analysis

Iterative relation (11) can be rewritten in the form

$$S_s^{-1} y_n^{k+1} = S_s^{-1} T_s y_n^k + g_n + S_s^{-1} p_{s,n}(k),$$

and by putting

$$\mathcal{C}_s = S_s^{-1}, \quad \mathcal{D}_s = S_s^{-1} T_s, \tag{13}$$

the iterative relation

$$\mathcal{C}_s y_n^{k+1} = \mathcal{D}_s y_n^k + g_n + \mathcal{C}_s p_{s,n}(k), \tag{14}$$

corresponding to the splitting $\mathcal{L} = \mathcal{C}_s - \mathcal{D}_s$, is generated.

Lemma 4.1. *Let $A_n = M_n - N_n - D_n$ be a composite M-splitting and $h > 0$. Then, $G_n \geq 0$, $L_n \geq 0$ and $\rho(G_n) < 1$, where the matrices G_n and L_n were defined in (8).*

Proof. The proof is similar to Lemma 8 in [7] and is omitted here. □

Note that, the assumption $\rho(G_n) < 1$ shows that the matrix $I - G_n$ is nonsingular and $\sum_{v=0}^{\infty} G_n^v = (I - G_n)^{-1}$. Therefore,

$$S_s = \sum_{v=0}^{s-1} G_n^v = (I - G_n^s)(I - G_n)^{-1}.$$

This shows that the matrix S_s is nonsingular. Since $T_s = G_n^s + \sum_{v=0}^{s-1} G_n^v L_n$, we have

$$T_s = G_n^s + S_s L_s = G_n^s + (I - G_n^s)(I - G_n)^{-1} L_n.$$

Similarly $\mathcal{C}_s$ and $\mathcal{D}_s$ can rewritten in the form

$$\begin{aligned} \mathcal{C}_s &= S_s^{-1} = (I - G_n)(I - G_n^s)^{-1}, \\ \mathcal{D}_s &= S_s^{-1} T_s = (I - G_n)(I - G_n^s)^{-1} G_n^s + L_n, \end{aligned} \tag{15}$$

and hence we can derive

$$\mathcal{L} = \mathcal{C}_s - \mathcal{D}_s = I - G_n - L_n. \tag{16}$$

Lemma 4.2. *If A_n is an M-matrix, $A_n = M_n - N_n - D_n$ is a composite M-splitting and $h > 0$, then $\mathcal{L} = I - G_n - L_n$ is inverse nonnegative, where A_n, M_n, N_n and D_n are approximations for $A(t_n)$, $M(t_n)$, $N(t_n)$ and $D(t_n)$, respectively, $n = 1, 2, \ldots, N$.*

Proof. The proof is similar to that of the Proposition 9 in [7] and is omitted here. □

Theorem 4.3. *Let A_n be an M-matrix and $A_n = M_n - N_n - D_n$, $n = 1, 2, \ldots, N$ a composite M-splitting. If $h > 0$ and $s \geq 1$ then $\rho(T_s) < 1$, and hence the TSWR method is convergent.*

Proof. Since the iterative method given by (11) is equivalent to (14) which is itself corresponding to the splitting $\mathcal{L} = \mathcal{C}_s - \mathcal{D}_s$, by the hypotheses and Lemma 4.2, we conclude that the matrix $\mathcal{L}$ is inverse nonnegative. Thus, from Lemma 3.2 we have $\rho(\mathcal{C}_s^{-1}\mathcal{D}_s) < 1$. On the other hand, $\mathcal{C}_s^{-1}\mathcal{D}_s = S_s S_s^{-1} T_s = T_s$. Therefore $\rho(T_s) < 1$. By invoking Lemma 3.4 we can deduce that the iterative method defined by Eq. (11) is convergent which implies that the TSWR method is convergent. □

Lemma 4.4. *([11]) Let $A = C_1 - D_1 = C_2 - D_2$ be two convergent weak regular splittings of A such that $C_2^{-1} \geq C_1^{-1}$ and let x_1 and x_2 be the nonnegative Perron-Frobenius eigenvectors of $T_1 = C_1^{-1}D_1$ and $T_2 = C_2^{-1}D_2$, respectively. If $D_2x_2 \geq 0$ or $D_1x_1 \geq 0$ with $x_1 > 0$, then $\rho(T_2) \leq \rho(T_1)$.*

Lemma 4.5. *Let $A = M - N - D$ be a composite M-splitting and $h > 0$. For $s \geq 1$ consider the convergence matrix $T_S = \mathcal{C}_s^{-1}\mathcal{D}_s$ corresponding to the weak regular splitting $\mathcal{L} = \mathcal{C}_s - \mathcal{D}_s$ as defined in (15) and (16). If x is the Perron-Frobenius eigenvector of $T_S = \mathcal{C}_s^{-1}\mathcal{D}_s$, then $\mathcal{D}_s x \geq 0$.*

Proof. The proof is quite similar to Lemma 14 in [7] and is omitted here. □

Although Theorem 4.3 guarantees convergence of the TSWR method independently of the number of inner iterations, but the next theorem indicates if the number of inner iterations increases then the spectral radius of the iteration matrix T_s decreases or remains unchanged. In this case the speed of the convergence of the TSWR method may increase or remain unchanged.

Theorem 4.6. *Let A_n be an M-matrix, $A_n = M_n - N_n - D_n$ a composite M-splitting and $h > 0$. Let us indicate by T_{s_1} and T_{s_2} the matrices of convergence of TSWR method with s_1 and s_2 inner iterations, respectively. If $1 \leq s_2 \leq s_1$, then $\rho(T_{s_1}) \leq \rho(T_{s_2}) < 1$.*

Proof. Consider the two couples of matrices $(\mathcal{C}_{s_1}, \mathcal{D}_{s_1})$ and $(\mathcal{C}_{s_2}, \mathcal{D}_{s_2})$ as defined in (15). By (10) and (13), we have

$$\mathcal{C}_{s_1}^{-1} = \sum_{v=0}^{s_1-1} G_n^v \quad \text{and} \quad \mathcal{C}_{s_2}^{-1} = \sum_{v=0}^{s_2-1} G_n^v.$$

Since $G_n = h(I + hM_n)^{-1}N_n \geq 0$ from $s_2 \leq s_1$, we derive

$$\sum_{v=0}^{s_2-1} G_n^v \leq \sum_{v=0}^{s_1-1} G_n^v \Rightarrow \mathcal{C}_{s_2}^{-1} \leq \mathcal{C}_{s_1}^{-1}.$$

By Lemma 4.5, $\mathcal{L} = \mathcal{C}_{s_1} - \mathcal{D}_{s_1}$ and $\mathcal{L} = \mathcal{C}_{s_2} - \mathcal{D}_{s_2}$ are both convergent weak regular splitting and, denoted with x_2 the Perron-Frobenius eigenvector of T_{s_2}, we have $\mathcal{D}_{s_2}x_2 \geq 0$. Hence by Lemma 4.4 we deduce that $\rho(T_{s_1}) \leq \rho(T_{s_2}) < 1$. $\square$

Under the hypothesis of Theorem 4.6, when $s \to \infty$, we obtain

$$\begin{aligned}
\mathcal{C}_{\infty}^{-1} &= \lim_{s\to\infty} \mathcal{C}_s^{-1} = \lim_{s\to\infty} \sum_{v=0}^{s-1} G_n^v = (I - G_n)^{-1}, \\
\mathcal{D}_{\infty} &= \mathcal{C}_{\infty} - \mathcal{L} = (I - G_n) - (I - G_n - L_n) = L_n, \\
T_{\infty} &= \lim_{s\to\infty} T_s = \lim_{s\to\infty} \mathcal{C}_s^{-1}\mathcal{D}_s = (I - G_n)^{-1}L_n.
\end{aligned}$$

Since

$$\begin{aligned}
(I - G_n)^{-1}L_n &= (I - h(I + hM_n)^{-1}M_n)^{-1}h(I + hM_n)^{-1}D_n \\
&= ((I + hM_n) - hN_n)^{-1}hD_n = h(I + hC_n)^{-1}D_n.
\end{aligned}$$

Therefore

$$T_{\infty} = (I + h(M_n - N_n))^{-1}hD_n, \tag{17}$$

as a result the TSWR method tends to the WR method.

Lemma 4.7. *Let $A_n = M_n - N_n - D_n$ be a composite M-splitting and $h_1, h_2 > 0$ such that $h_1 \leq h_2$. If $G_{n_1}, G_{n_2}, L_{n_1}, L_{n_2}$ are the matrices defined in* (8) *with $h = h_1$ and h_2, respectively, then $G_{n_1} \leq G_{n_2}$ and $L_{n_1} \leq L_{n_2}$.*

Proof. The proof immediately follows from Lemma 16 in ([7]). $\square$

The next theorem shows that by decreasing the step size h, spectral radius of the convergence matrix T_s may decrease or remain unchanged and as a result the speed of the convergence of the TSWR method may increase.

Theorem 4.8. *Let A_n be an M-matrix, $A_n = M_n - N_n - D_n$ a composite M-splitting, and $h_1, h_2 > 0$ such that $h_1 \leq h_2$. Let us indicate by $T_s^{(1)}$ and $T_s^{(2)}$ the matrices of convergence of TSWR method with $h = h_1$ and $h = h_2$ respectively and the same fixed number $s \geq 1$ of inner iterations. Then, $\rho(T_s^{(1)}) \leq \rho(T_s^{(2)}) < 1$.*

Proof. From (10) we know that

$$T_s^{(1)} = G_{n_1}^s + \sum_{v=0}^{s-1} G_{n_1}^v L_{n_1}.$$

Let $x > 0$ be the Perron-Frobenius eigenvector of $T_s^{(1)}$. Without loss of generality we can assume that $\|x\|_2 = 1$. By Lemma 4.7, we have

$$T_s^{(1)}x = G_{n_1}^s x + \sum_{v=0}^{s-1} G_{n_1}^v L_{n_1} x \leq G_{n_2}^s x + \sum_{v=0}^{s-1} G_{n_2}^v L_{n_2} x = T_s^{(2)} x.$$

Therefore,

$$\rho(T_s^{(1)})x \leq \rho(T_s^{(2)})x.$$

Hence, $\rho(T_s^{(1)}) \leq \rho(T_s^{(2)}) < 1$, which completes the proof. $\square$

5. Numerical results

In this section, we give some numerical experiments to validate the theoretical results.

Example 5.1. In IVP (1) assume that $A(t) : [t_0, T] \rightarrow \mathbb{R}^{pq \times pq}$ is defined as

$$A(t) = \begin{pmatrix} \widetilde{A}(t) & -4t^2 I_p & & & \\ -4t^2 I_p & \widetilde{A}(t) & -4t^2 I_p & & \\ & -4t^2 I_p & \widetilde{A}(t) & -4t^2 I_p & \\ & & \ddots & \ddots & \ddots \\ & & & -4t^2 I_p & \widetilde{A}(t) \end{pmatrix},$$

where I_p indicates the $p \times p$ identity matrix and $\widetilde{A}(t) : [t_0, T] \rightarrow \mathbb{R}^{p \times p}$, is defined

$$\widetilde{A}(t) = \begin{pmatrix} 20t & -4t^2 & & & \\ -4t^2 & 20t & -4t^2 & & \\ & -4t^2 & 20t & -4t^2 & \\ & & \ddots & \ddots & \ddots \\ & & & -4t^2 & 20t \end{pmatrix}.$$

The function $f(t)$ is computed such that the exact solution is given by

$$y(t) = \left(\frac{t}{2}, \frac{t^2}{4}, \frac{t^3}{8}, \frac{t^4}{16}, \frac{t^5}{32}, \ldots, \frac{t}{2}, \frac{t^2}{4}, \frac{t^3}{8}, \frac{t^4}{16}, \frac{t^5}{32}\right)^T \in \mathbb{R}^{pq}.$$

We set $t_0 = 0$, $T = 1$ and $p = q = 5$. By applying the Kronecker product [1], matrix $A(t)$ can be rewritten in the form:

$$A(t) = 20t(I_5 \otimes I_5) - [4t^2(I_5 \otimes R_5) + 4t^2(R_5 \otimes I_5)],$$

where R_5 is a 5×5 tridiagonal matrix with ones on the first diagonal above and below the main diagonal and zeros elsewhere. By putting $B(t) = 4t^2(I_5 \otimes R_5) + 4t^2(R_5 \otimes I_5)$, and from analysis of eigenvalues of R_5, we have

$$\rho(B(t)) = 16t^2 \cos(\frac{\pi}{6}).$$

Table 1: Values of $\rho(T_s)$ for TSWR method for Example 1.

s	$h=0.1$	$h=0.3$	$h=0.5$	$h=0.8$	$h=1$
1	0.3410808	0.4586948	0.4926722	0.5140928	0.5216529
2	0.2287084	0.3345479	0.3676991	0.3891921	0.3968874
3	0.2095444	0.3060751	0.3369137	0.3570868	0.3643452
4	0.2062762	0.2995450	0.3293301	0.3488342	0.3558574
5	0.2057188	0.2980473	0.3274620	0.3467130	0.3536435
6	0.2056238	0.2977038	0.3270018	0.3461677	0.3530661
7	0.2056075	0.2976250	0.3268885	0.3460275	0.3529155
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	0.2056042	0.2976016	0.3268514	0.3459790	0.3528623

Since $t \in [0,1]$, we have $20t \geq 16t^2 \cos(\frac{\pi}{6})$. This shows that the matrix $A(t)$ can be written in the form $A(t) = s(t)I - B(t)$, such that $\rho(B(t)) \leq s(t)$ for $t \in [0,1]$. Hence, $A(t)$ is an M-matrix for every $t \in [0,1]$. Assuming

$$M(t) = 20t(I_5 \otimes I_5), \quad D(t) = 4t^2(R_5 \otimes I_5) \quad \text{and} \quad N(t) = 4t^2(I_5 \otimes R_5),$$

it is easy to verify that $A(t) = M(t) - N(t) - D(t)$ is a composite M-splitting for every $t \in [0,1]$.

According to Theorems 4.6 and 4.8, numerical results given in Table 1 indicate the monotonicity of $\rho(T_s)$ at varying of s and h, at $t = 0.8$. The values of $\rho(T_s)$ as $s \to \infty$ have been computed by using (17). We mention, here, that both of the outer and inner splittings of the TSWR method is the block Jacobi type.

In continuation, we compare the numerical results of the WR method given by (3) and the TSWR method presented by (5). For the two methods, we set $h = 0.1$, $N = 10$ and all of our computations terminate once the current iterations obey $\|y_n^{k+1} - y_n^k\|_\infty \leq 10^{-3}$, $n = 0, 1, \ldots, 1/h$ or $k > 1000$. In the TSWR method the number of the inner iterations is set to be $s = 5$.

Some numerical results are given in Figure 1. This figure shows that the WR and the TSWR provide almost the same solution with the common stop condition for outer iterations. Figure 2 shows that the TSWR method approximates the exact solution in a better way than those provided by the WR method. The number of outer iterations is 14 for the TSWR method but it is 20 for the WR method. In fact, all of the above comparisons show the advantages of using inner iterations in WR method to make TSWR method.

To investigate the effect of the number of inner iterations s and the step size h on the speed of the convergence of the TSWR method we have provided Table 2. Numerical results given in this table show that by decreasing the step size the number of outer iterations reduces and the TSWR method converges rapidly. We also observe that by increasing the number of inner iterations s the number of outer iterations decreases or remains unchanged.

Example 5.2. Consider the previous example with a splitting of the form

$$M(t) = I_5 \otimes (4t^2 F_5 + 20t I_5), \quad D(t) = 4t^2(R_5 \otimes I_5),$$

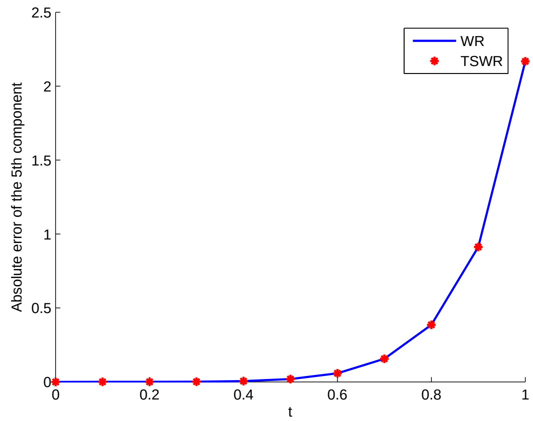

Figure 1: The absolute error for the fifth component of y for WR and TSWR with Jacobi/Jacobi splittings.

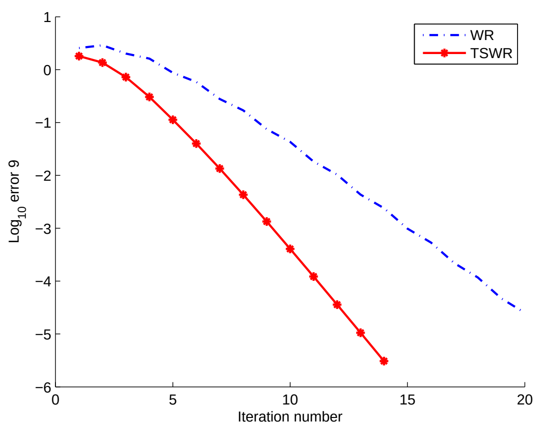

Figure 2: Log_{10} of error$_9$ where error$_9 = \|y_9^{k+1} - y_9^k\|_\infty$ of the iterate number at $t_9 = t_0 + 9h = 0.9$ for WR and TSWR with Jacobi/Jacobi splittings.

Table 2: Number of outer iterations versus s and h for the TSWR method for Example 1.

h	$s = 2$	$s = 3$	$s = 5$	$s = 8$	$s = 10$	$s = 15$	$s = 20$
0.1	18	15	14	14	14	14	14
0.2	182	134	91	61	51	35	27
0.5	283	214	154	112	95	70	55
1	1000	848	701	641	628	618	617

Table 3: Values of $\rho(T_s)$ for the TSWR method for Example 2.

s	$h = 0.1$	$h = 0.3$	$h = 0.5$	$h = 0.8$	$h = 1$
1	0.3124548	0.4369381	0.4737154	0.4970554	0.5053195
2	0.2085735	0.2765793	0.2958105	0.3078366	0.3120629
3	0.2008186	0.2643622	0.2823397	0.2936032	0.2975668
4	0.2004802	0.2632316	0.2808238	0.2917967	0.2956482
5	0.2004274	0.2630811	0.2806321	0.2915759	0.2954165
6	0.2004231	0.2630563	0.2805941	0.2915269	0.2953631
7	0.2004230	0.2630554	0.2805925	0.2915248	0.2953607
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	0.2004230	0.2630552	0.2805922	0.2915244	0.2953602

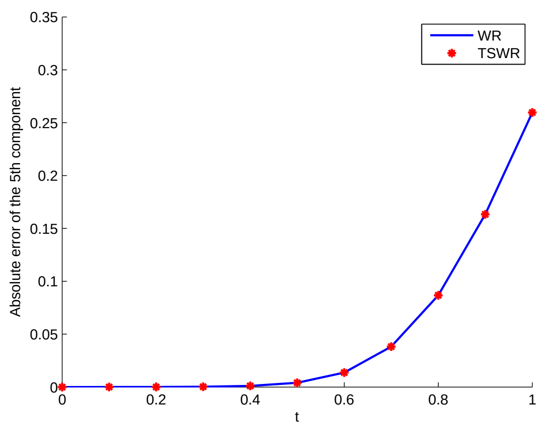

Figure 3: The absolute error for the fifth component of y for WR and TSWR with Jacobi/Gauss-Seidel splittings.

$$N(t) = 4t^2(I_5 \otimes (R_5 - F_5)),$$

where F_5 is the lower part of R_5. By the above matrices, outer splitting of the TSWR method would be Jacobi type and inner splitting would be Gauss-Seidel type. It is easy to verify that $A(t) = M(t) - N(t) - D(t)$ is a composite M-splitting for $t \in [0, 1]$. In Table 3 the values of the spectral radius of T_s versus h and s have been presented. Numerical results given in this table indicate the monotonicity of $\rho(T_s)$ at varying of s and h, at $t = 0.9$.

Similar to the previous example Figures 3 and 4 are provided. These figures not only show the superiority of the TSWR method to the WR method but also the superiority of the TSWR method produced by the Gauss-Seidel method instead of the the Jacobi as the inner iterations.

Similar to the previous example we have provided Table 4. As seen, this table shows that by decreasing the step size the number of outer iterations decreased and the TSWR method converges rapidly. We also observe that by increasing the number of inner iterations s the number of outer iterations reduces or remains unchanged.

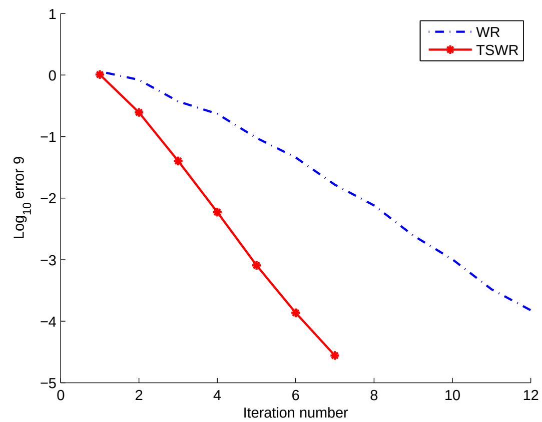

Figure 4: Log_{10} error_9 where $\text{error}_9 = \|y_9^{k+1} - y_9^k\|_\infty$ of the iterate number at $t_9 = t_0 + 9h = 0.9$ for WR and TSWR with Jacobi/Gauss-Seidel splittings.

Table 4: Number of outer iterations versus s and h for the TSWR method for Example 2.

h	$s=2$	$s=3$	$s=5$	$s=8$	$s=10$	$s=15$	$s=20$
0.1	9	7	5	5	5	5	5
0.2	102	27	27	27	27	24	22
0.5	313	221	194	184	146	137	119
1	846	667	594	590	586	576	543

6. Conclusion

We have studied the WR and the TSWR method for initial value problems with non-constant coefficients. Under some specific conditions, we have analyzed convergence of the methods and compared their speed of convergence. Numerical results show that the TSWR method in general is superior to the WR method.

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