Convergence of the variational iteration method for the cubic nonlinear Schrödinger equation

Davod Khojasteh Salkuyeh*, Mehdi Bastani

1 Department of Mathematics, University of Mohaghegh Ardabili, P.O.Box. 179, Ardabil, Iran.

Abstract. In this paper, the variational iteration method (VIM) is applied to solve the cubic nonlinear Schrödinger (CNLS) equation. Some convergence results of VIM for solving CNLS equation are also given. Two examples are given to illustrate the theoretical results.

Keywords: Variational iteration method; Cubic nonlinear Schrödinger; Convergence; Lagrange multipliers; Correctional functional.

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1 Introduction

The cubic nonlinear Schrödinger (CNLS) equation is one of the most important equations of mathematical physics with applications in many different fields such as plasma physics, nonlinear optics, water waves, and bimolecular dynamics [2–5,19]. The one-dimensional (1D) CNLS equation can be written as

\[
\begin{align*}
    iu_t + u_{xx} + q |u|^2 u &= 0, \quad x \in \Omega, \quad t > 0, \\
    u(x,0) &= g(x), \quad x \in \Omega
\end{align*}
\]  

(1.1)

where \( \Omega = [\ell_0, \ell_1] \), \( q \geq 0 \) and \( i = \sqrt{-1} \). Similarly, the two-dimensional (2D) CNLS equation may be expressed as

\[
\begin{align*}
    iu_t + \nabla^2 u + q |u|^2 u &= 0, \quad (x,y) \in \Omega, \quad t > 0 \\
    u(x,y,0) &= f(x,y), \quad (x,y) \in \Omega
\end{align*}
\]  

(1.2)
where $\Omega = [\ell_0, \ell_1] \times [\ell_0, \ell_1]$ and $\nabla^2 = \partial^2/\partial^2 x + \partial^2/\partial^2 y$ is the Laplace operator.

Several papers have been presented to investigate the solution of the CNLS equation in the literature. Bratsos et al. in [1] and Khuri in [16], applied the Adomian decomposition to solve the CNLS equation. Khani et al. in [15] used the exp-function method to find the solutions of the CNLS equation. Wazwaz applied the sine-cosine and tanh methods to derive some exact solution of the CNLS equation in [23]. The spectral collocation method with preconditioning has been applied to solve the CNLS equation by Javidi and Golbabai [13]. In [14], the authors investigated the solution of the CNLS equation by the differential transform method. Tian in [22] proposed a high-order compact ADI method for solving unsteady 2D Schrödinger equation.

As we know, the variational iteration method (VIM) presented by He [6–11] is a powerful mathematical tool for finding solutions of linear and nonlinear problems and it can be implemented easily in practice. It has been successfully applied for solving various PDEs and ODEs [6–11]. Sweilam in [20] and Wazwaz [24] applied the VIM for computing an approximate solution to the 1D and 2D cubic nonlinear Schrödinger (CNLS) equations. There are a few papers that verify the convergence of the VIM (see for example [17,18,21]). In this paper, we use the VIM to solve problems (1.1) and (1.2) and our emphasis is on verifying the convergence of the proposed method.

This paper is organized as follows. In section 2, a brief description of the VIM is given. Section 3 is devoted to the convergence of the VIM for the CNLS equation. Two illustrative examples are given in section 4. Some concluding remarks are presented in section 5.

2 A brief description of the VIM

The variational iteration method transforms the differential equation to a recurrence sequence of functions and the limit of the sequence, if exists, is considered as the solution of the differential equation. Consider the following differential equation

$$Lu(t) + Nu(t) = g(t),$$

where $L$ is a linear operator, $N$ a nonlinear operator and $g(t)$ is an inhomogeneous term. In the VIM, a correctional functional as

$$u_{m+1}(t) = u_m(t) + \int_0^t \lambda (Lu_m(s) + Nu_m(s) - g(s)) ds, \quad m = 0, 1, 2, \ldots,$$

is made, where $\lambda$ is a general Lagrangian multiplier [12] which can be identified optimally via the variational theory. Obviously the successive approximations $u_j$, $j = 0, 1, \ldots$, can be computed by determining $\lambda$. Here, the function $\tilde{u}_m$ is a restricted variation which means $\delta \tilde{u}_m = 0$. 

3 Convergence of the VIM for the CNLS equation

We first consider the 1D CNLS equation. Let
\[ u = u(x, t) = v(x, t) + iw(x, t), \] (3.1)
and
\[ u(x, 0) = g(x) = g_R(x) + ig_I(x), \quad \ell_0 \leq x \leq \ell_1. \] (3.2)
Substituting (3.1) and (3.2) in (1.1) results in the following coupled system of partial differential equations
\[
\begin{align*}
  v_t + w_{xx} + q(v^2 + w^2)w &= 0, \\
  w_t - v_{xx} - q(v^2 + w^2)v &= 0,
\end{align*}
\] (3.3)
with the following initial conditions
\[ v(x, 0) = g_R(x), \quad w(x, 0) = g_I(x). \]

By using the variational iteration method, we construct the following correction functionals
\[
\begin{align*}
  v_{n+1}(x, t) &= v_n(x, t) + \int_0^t \lambda_1(\tau)[v_n\tau + \dot{w}_{nxx} + q(\dot{v}_n^2 + \dot{w}_n^2)\dot{w}_n]d\tau, \\
  w_{n+1}(x, t) &= w_n(x, t) + \int_0^t \lambda_2(\tau)[w_n\tau - \dot{v}_{nxx} - q(\dot{v}_n^2 + \dot{w}_n^2)\dot{v}_n]d\tau,
\end{align*}
\] (3.4)
where \( \lambda_1 \) and \( \lambda_2 \) are general Lagrange multipliers, \( \dot{v}_{nxx} = \frac{\partial^2}{\partial x^2} \dot{v}_n, \dot{v}_n, \dot{w}_{nxx} = \frac{\partial^2}{\partial x^2} \dot{w}_n \) and \( \dot{w}_n \) denote restricted variations, i.e. \( \delta \dot{v}_{nxx} = \delta \dot{v}_n = \delta \dot{w}_{nxx} = \delta \dot{w}_n = 0 \). Then, making the correction functionals stationary, we obtain the following stationary conditions:
\[
\begin{align*}
  1 + \lambda_1'(\tau)|_{\tau = t} &= 0, \quad \lambda_1'(\tau)|_{\tau = t} = 0, \\
  1 + \lambda_2'(\tau)|_{\tau = t} &= 0, \quad \lambda_2'(\tau)|_{\tau = t} = 0,
\end{align*}
\]
Therefore the Lagrange multipliers can be identified as
\[ \lambda_1(\tau) = -1, \quad \lambda_2(\tau) = -1. \]

Hence, Eq. (3.4) can be written as
\[
\begin{align*}
  v_{n+1}(x, t) &= v_n(x, t) - \int_0^t [v_n\tau + w_{nxx} + q(v_n^2 + w_n^2)w_n]d\tau, \\
  w_{n+1}(x, t) &= w_n(x, t) - \int_0^t [w_n\tau - v_{nxx} - q(v_n^2 + w_n^2)v_n]d\tau,
\end{align*}
\] (3.5)
This recursive formula has been given by Sweilam in [20] without any discussion about the convergence of this method. We now state the next theorem concerning its convergence.
Theorem 3.1. Let $\Omega^* = \Omega \times [0, T]$, $v, w \in C^2(\Omega^*)$ and the initial guess be the initial condition. Let also $v_n, w_n \in C^2(\Omega^*)$ and be two uniformly bounded sequences on $\Omega^*$. If there exist two constants $c_v$ and $c_w$ such that

\[
\begin{aligned}
|\frac{\partial^2}{\partial x^2}(v_n - v)| &\leq c_v|v_n - v|, \\
|\frac{\partial^2}{\partial x^2}(w_n - w)| &\leq c_w|w_n - w|,
\end{aligned}
\]

for $n = 0, 1, 2, \ldots$, then $v_n$ and $w_n$ defined by (3.5) converge to $v$ and $w$, respectively.

Proof. It is easy to verify that the first equation of (3.5) can be written as

\[ v_{n+1}(x, t) = v_n(x, 0) - \int_0^t [w_{nxx} + q(v_n^2 + w_n^2)w_n]d\tau, \tag{3.6} \]

From this equation we have $v_{n+1}(x, 0) = v_n(x, 0)$. Therefore, $v_j(x, 0) = v(x, 0)$ for $j \geq 0$. On the other hand the first equation of (3.3) may be written as

\[ v(x, t) = v(x, 0) - \int_0^t [w_{xx} + q(v^2 + w^2)]d\tau, \tag{3.7} \]

Now, from Eqs. (3.6) and (3.7) we get

\[ E_{n+1} = -\int_0^t [\frac{\partial^2}{\partial x^2}F_n + q(v_n^2 + w_n^2)w_n - q(v^2 + w^2)w]d\tau, \]

where $E_j = E_j(x, t) = v_j(x, t) - v(x, t)$ and $F_j = F_j(x, t) = w_j(x, t) - w(x, t)$. Now we proceed as follows

\[
\begin{aligned}
E_{n+1} &= -\int_0^t [\frac{\partial^2}{\partial x^2}F_n + q(v_n^2 + w_n^2)w_n - q(v_n^2 + w_n^2)w_n + q(v_n^2 + w_n^2)w - q(v^2 + w^2)w]d\tau, \\
&= -\int_0^t [\frac{\partial^2}{\partial x^2}F_n + q(v_n^2 + w_n^2)F_n + qw((v_n + v)E_n + (w_n + w)F_n)]d\tau, \\
&= -\int_0^t [\frac{\partial^2}{\partial x^2}F_n + q(v_n^2 + w_n^2 + w^2 + w_n)F_n + qw(v_n + v)E_n]d\tau.
\end{aligned}
\]

Since $v_n, w_n, w, v \in C^2(\Omega^*)$ and $v_n$ and $w_n$ are uniformly bounded on $\Omega^*$, there exist constants $M_v, M_w, N_v$ and $N_w$ such that $|v_n| \leq M_v, |w_n| \leq M_w, |v| \leq N_v$ and $|w| \leq N_w$ on $\Omega^*$. Therefore, we have

\[
\begin{aligned}
|E_{n+1}| &\leq \int_0^t [\frac{\partial^2}{\partial x^2}F_n] + q(|v_n|^2 + |w_n|^2 + |w|^2 + |w_n||w|)|F_n| + q|w|(|v_n| + |v|)|E_n|]d\tau \\
&\leq \int_0^t [c_v|F_n| + q(M_v^2 + M_w^2 + N_v^2 + M_wN_w)|F_n| + qN_w(M_v + N_v)|E_n|]d\tau \\
&= \int_0^t (c_{11}|E_n| + c_{12}|F_n|)d\tau,
\end{aligned}
\]
where
\[ c_{11} = qN_w(M_v + N_v), \]
\[ c_{12} = c_v + q(M_v^2 + M_w^2 + N_v^2 + M_wN_w). \]

Now, by using the Cauchy-Schwartz inequality we get
\[ |E_{n+1}| \leq \int_0^t (c_{11}|E_n| + c_{12}|F_n|)d\tau \]
\[ \leq \sqrt{c_{11}^2 + c_{12}^2} \int_0^t \sqrt{|E_n|^2 + |F_n|^2}d\tau \]
\[ = c_1 \int_0^t \sqrt{|E_n|^2 + |F_n|^2}d\tau, \tag{3.8} \]

where \( c_1 = \sqrt{c_{11}^2 + c_{12}^2}. \) In the same manner, one can see that there is a positive constant \( c_2 \) such that
\[ |F_{n+1}| \leq c_2 \int_0^t \sqrt{|E_n|^2 + |F_n|^2}d\tau, \tag{3.9} \]

Now from Eqs. (3.8) and (3.9) we conclude that
\[ \sqrt{|E_{n+1}|^2 + |F_{n+1}|^2} \leq \sqrt{c_1^2 + c_2^2} \int_0^t \sqrt{|E_n|^2 + |F_n|^2}d\tau. \tag{3.10} \]

Let \( S_j = \sqrt{|E_j|^2 + |F_j|^2}, \ j = 0, 1, \ldots, \) and \( c = \sqrt{c_1^2 + c_2^2}. \) In this case, from (3.10) we obtain
\[ S_{n+1}(x, t) \leq c \int_0^t S_n(x, \tau)d\tau \]

Now, letting \( M = \max_{(x,t) \in \Omega^*} S_0(x, t), \) we proceed as follows
\[ S_1(x, t) \leq c \int_0^t S_0(x, \tau)d\tau \leq M \int_0^t d\tau = Mt, \]
\[ S_2(x, t) \leq c \int_0^t S_1(x, \tau)d\tau \leq c \int_0^t cM \tau d\tau = c^2 M \frac{t^2}{2}, \]
\[ S_3(x, t) \leq c \int_0^t S_2(x, \tau)d\tau \leq c \int_0^t c^2 M \frac{\tau^2}{2} d\tau = c^3 M \frac{t^3}{3!}, \]
\[ \vdots \]
\[ S_n(x, t) \leq c \int_0^t S_{n-1}(x, \tau)d\tau \leq c \int_0^t c^{n-1} M \frac{\tau^{n-1}}{(n-1)!} d\tau = c^n M \frac{t^n}{n!}. \]

Therefore
\[ S_n(x, t) \leq M \frac{(ct)^n}{n!} \leq M \frac{(cT)^n}{n!} \to 0, \]
as \( n \) tends to infinity and this completes the proof.
Now we turn to the 2D CNLS equation (1.2). Similar to the one-dimensional case the recursive formula can be written

\begin{align*}
  v_{n+1}(x, y, t) &= v_n(x, y, t) - \int_0^t \left[ v_{n\tau} + \nabla^2 w_n + q(v_n^2 + w_n^2)w_n \right] d\tau, \\
  w_{n+1}(x, y, t) &= w_n(x, y, t) - \int_0^t \left[ w_{n\tau} - \nabla^2 v_n - q(v_n^2 + w_n^2)v_n \right] d\tau.
\end{align*}

(3.11)

We now state the following theorem.

**Theorem 3.2.** Let \( \Omega^* = \Omega \times [0, T] \), \( v, w \in C^2(\Omega^*) \) and the initial guess be the initial condition. Let also \( v_n, w_n \in C^2(\Omega^*) \) and be two uniformly bounded sequences on \( \Omega^* \). If there exist constants \( c_{v_x}, c_{v_y}, c_{w_x} \), and \( c_{w_y} \) such that

\begin{align*}
  |\frac{\partial^2}{\partial x^2}(v_n - v)| &\leq c_{v_x}|v_n - v|, & |\frac{\partial^2}{\partial x^2}(w_n - w)| &\leq c_{w_x}|w_n - w|, \\
  |\frac{\partial^2}{\partial y^2}(v_n - v)| &\leq c_{v_y}|v_n - v|, & |\frac{\partial^2}{\partial y^2}(w_n - w)| &\leq c_{w_y}|w_n - w|,
\end{align*}

for \( n = 0, 1, 2, \ldots \), then \( v_n \) and \( w_n \) defined by (3.11) converge to \( v \) and \( w \), respectively.

**Proof.** The proof of this theorem is quite similar to Theorem 1 and is omitted here. \( \square \)

### 4 Illustrative examples

In this section, we present two examples to illustrate the theoretical results presented in the previous section.

**Example 4.1.** Consider Eq. (1.1) with \( q = 2 \) and \( \Omega^* = [-\pi, \pi] \times [0, 1] \). The analytical solution of this problem is (see [23])

\[ u(x, t) = e^{i(x+t)}. \]

Therefore,

\[ v(x, t) = \cos(x + t), \]
\[ w(x, t) = \sin(x + t), \]

and we use

\[ v_0(x, t) = v(x, 0) = \cos x, \]
\[ w_0(x, t) = w(x, 0) = \sin x, \]

as our initial guess. We use \( n = 1, 2, 3, 4, 5 \) iterations of the VIM for solving the problem. In Figure 1 the absolute values of \( v - v_n \) and \( w - w_n \) at \( t = 0.3 \) for \( n = 1, 2, 3, 4, 5 \) are displayed. As we observe, this figure shows the convergence of the VIM for solving the
Figure 1: Convergence of the VIM for Example 1. Absolute value of \( v - v_n \) (left) and \( w - w_n \) (right) for \( n = 1, 2, 3, 4, 5 \) at \( t = 0.3 \).

problem. For more investigation, in Table 1, we report the values of \( u_5, w_5, v \) and \( w \) in some points. In the last column of this table the absolute errors of the computed solution are given. As we observe there is a good agreement between the exact and computed solutions.

**Example 4.2.** Our second example is devoted to 2D CNLS equation in \( \Omega^* = [-\pi, \pi] \times [-\pi, \pi] \times [0, 1] \)

\[ iu_t + \nabla^2 u + 3u|u|^2 = 0, \]

with the analytical solution

\[ u(x, y, t) = e^{i(x+y+t)}, \]

i.e,

\[ v(x, y, t) = \cos(x + y + t), \]
\[ w(x, y, t) = \sin(x + y + t). \]

Therefore, we use

\[ v_0(x, y, t) = v(x, y, 0) = \cos(x + y), \]
\[ w_0(x, y, t) = w(x, y, 0) = \sin(x + y), \]

as the initial guess. Similar to the previous example, we use \( n = 1, 2, 3, 4, 5 \) iterations of the VIM for solving the problem. In Figure 2 the graph for the absolute values of \( v - v_n \) and \( w - w_n \) at \( t = 0.3 \) for \( n = 1, 2, 3, 4, 5 \) are displayed. As we observe, this figure shows
Table 1: Comparison between the exact and the computed solution with \( n = 5 \) at \( t = 0.3 \) for the Example 1.

| \( x \)     | \( v \)    | \( v_5 \) | \( |v - v_5| \) |
|------------|------------|-----------|-----------------|
| \( -\pi \) | -0.95533649 | -0.95536439 | 2.79E-05         |
| \( -2\pi/3 \) | -0.22174024 | -0.22176073 | 2.05E-05         |
| \( -\pi/3 \) | 0.73359625  | 0.73360366  | 7.41E-06         |
| 0          | 0.95533649  | 0.95536439  | 2.79E-05         |
| \( \pi/3 \) | 0.22174024  | 0.22176073  | 2.05E-05         |
| \( 2\pi/3 \) | -0.73359625 | -0.73360366 | 7.41E-06         |
| \( \pi \)  | -0.95533649 | -0.95536439 | 2.79E-05         |

| \( x \)     | \( w \)    | \( w_5 \) | \( |w - w_5| \) |
|------------|------------|-----------|-----------------|
| \( -\pi \) | -0.29552021 | -0.29551265 | 7.56E-06         |
| \( -2\pi/3 \) | -0.97510577 | -0.97512616 | 2.04E-05         |
| \( -\pi/3 \) | -0.67958557 | -0.67961351 | 2.79E-05         |
| 0          | 0.29552021  | 0.29551265  | 7.56E-06         |
| \( \pi/3 \) | 0.97510577  | 0.97512616  | 2.04E-05         |
| \( 2\pi/3 \) | 0.67958557  | 0.67961351  | 2.79E-05         |
| \( \pi \)  | -0.29552021 | -0.29551265 | 7.56E-06         |

the convergence of the VIM for solving the problem. In Table 2, the values of \( v_5, w_5, v \) and \( w \) in some points are presented. The absolute error of the computed solution are given in the last column of this table. As we observe there is a good agreement between the exact and computed solutions.

Figure 2: Convergence of the VIM for Example 2. Absolute value of \( v - v_n \) (left) and \( w - w_n \) (right) for \( n = 1, 2, 3, 4, 5 \) at \( t = 0.3 \).
Table 2: Comparison between the exact and the computed solution with $n = 5$ at $t = 0.3$ for the Example 2.

| $x$ | $v$   | $v_5$   | $|v - v_5|$ |
|-----|-------|---------|----------|
| $-\pi$ | -0.26749883 | -0.26754944 | 5.06E-05 |
| $-2\pi/3$ | 0.70071645 | 0.70072164 | 5.19E-06 |
| $-\pi/3$ | 0.96821528 | 0.96827108 | 5.58E-05 |
| 0 | 0.26749883 | 0.26754944 | 5.06E-05 |
| $\pi/3$ | -0.70071645 | -0.70072164 | 5.19E-06 |
| $2\pi/3$ | -0.96821528 | -0.96827108 | 5.58E-05 |
| $\pi$ | -0.26749883 | -0.26754944 | 5.06E-05 |

| $x$ | $w$   | $w_5$   | $|w - w_5|$ |
|-----|-------|---------|----------|
| $-\pi$ | -0.96355819 | -0.96359340 | 3.52E-05 |
| $-2\pi/3$ | -0.71343987 | -0.71350131 | 6.14E-05 |
| $-\pi/3$ | 0.25011831 | 0.25009209 | 2.62E-05 |
| 0 | 0.96355819 | 0.96359340 | 3.52E-05 |
| $\pi/3$ | 0.71343987 | 0.71350131 | 6.14E-05 |
| $2\pi/3$ | -0.25011831 | -0.25009209 | 2.62E-05 |
| $\pi$ | -0.96355819 | -0.96359340 | 3.52E-05 |

5 Conclusion

In this article, we applied the variational iteration method to solve the CNLS equation. We have demonstrated that under some mild conditions the variational iteration method converges for the CNLS equation. We have presented two illustrate examples which show the efficiency of the method.

References


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