Solution of the Complex Modified Korteweg-de Vries Equation by the Projected Differential Transform Method

Davod Khojasteh Salkuyeh[†1] and Mehdi Bastani[§]

[†] *Faculty of Mathematical Sciences, University of Guilan, P.O. Box 1914, Rasht, Iran*
e-mail: khojasteh@guilan.ac.ir, salkuyeh@gmail.com,

[§] *Department of Mathematics, University of Mohaghegh Ardabili,*
P.O.Box. 179, Ardabil, Iran
emails: bastani.mehdi@yahoo.com

Abstract

In this paper, we first transform the Complex Modified Korteweg-de Vries (CMKdV) equation into a real-valued system of partial differential equations. Then the projected differential transformation method is used to compute an approximate solution of the CMKdV equation. Three illustrative examples are given to show the effectiveness of the method and comparing with the differential transformation method and the variational iteration method.

Key words: Taylor expansion series, differential transformation method, projected, variational iteration method, CMKdV.
2000 MSC: 35E15, 40A30.

---

1. Introduction

The nonlinear phenomena play a crucial rule in a variety of scientific fields such as fluid mechanics, optical fibers, solid state physics, chemical kinetics and geochemistry and are modeled by nonlinear partial differential equations (NPDEs) [5, 19, 20, 21]. Computing exact solutions of NPDEs is of considerable importance, because, exact solutions help one to well understand the mechanism of the complicated physical phenomena and dynamical processes modeled by these NPDEs. Recently, several powerful mathematical methods such as the variational iteration method (VIM) [9, 10], Adomian decomposition method [3], Exp-function method (see [17] and references therein), differential transformation method (DTM) [22] and its projected version (PDTM) [13], and others have been presented for computing exact and approximate analytic solutions for nonlinear problems.

---

[1]Corresponding author

Mathematical modeling of many phenomena such as nonlinear evolution of plasma waves [14], propagation of transverse waves in a molecular chain [7] and in a generalized elastic solid [6] are represented by the Complex Modified Korteweg-de Vries (CMKdV) equation. There are several forms of the CMKdV equation in the literature. In this paper, we consider the following form of the CMKdV equation:

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial^3 u(x,t)}{\partial x^3} - \beta \frac{\partial(|u(x,t)|^2 u(x,t))}{\partial x}, \quad -\infty < x < \infty, \ t > 0. \tag{1}$$

where $u(x,t)$ is a complex-valued of the spatial coordinate x and the time t, and β is a real parameter. In [16], Muslu and Erbay numerically solved Eq. (1) by three different split-step Fourier schemes. A collocation method was used to solve the CMKdV equation by Ismail in [11]. He also used the Petrov-Galerkin method to solve the CMKdV equation in [12]. In [15], Korkmaz and Dag solved numerically the CMKdV equation using differential quadrature method based on cosine expansion. In this paper, we apply the PDTM method to solve the CMKdV equation.

This paper is organized as follows. In Section 2, a brief description of the DTM and PDTM is reviewed. Section 3 is devoted to the application of the PDTM to solve the CMKdV equation. Three illustrative examples are given in section 4. Finally, some concluding remarks are given in Section 5.

2. A brief description of DTM and PDTM

Consider the initial value problem

$$\mathcal{L}u(\mathbf{x},t) = f(\mathbf{x},t,u(\mathbf{x},t)), \quad \mathbf{x} \in \Omega, \quad t > 0, \tag{2}$$

where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathcal{L} = \partial/\partial t$ with appropriate initial and boundary conditions. Let $u(\mathbf{x},t)$ be analytic at $(\tilde{\mathbf{x}}, \tilde{t})$. In this case, the solution $u(\mathbf{x},t)$ to Eq. (2) can be represented by the Taylor expansion series

$$u(\mathbf{x},t) = \sum_{r_1=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \sum_{k=0}^{\infty} U(\mathbf{r},k) \left(\prod_{i=1}^{n} (x_i - \tilde{x}_i)^{r_i} \right) (t - \tilde{t})^k, \tag{3}$$

where

$$U(\mathbf{r},k) = \frac{1}{r_1! \cdots r_n! k!} \left[\frac{\partial^{r_1+\cdots+r_n+k} u(\tilde{\mathbf{x}}, \tilde{t})}{\partial x_1^{r_1} \cdots \partial x_n^{r_n} \partial t^k} \right].$$

$U(\mathbf{r},k)$ is called the $(n+1)$-dimensional differential transform of $u(\mathbf{x},t)$ at $(\tilde{\mathbf{x}}, \tilde{t})$ and is denoted by $U(\mathbf{r},k) = \mathcal{D}\{u(\mathbf{x},t)\}$ where $\mathcal{D}$ is the differential transform operator [13]. The inverse differential transform of $U(\mathbf{r},k)$ is defined by $u(\mathbf{x},t)$ of the form in (3) and is denoted by $u(\mathbf{x},t) = \mathcal{D}^{-1}\{U(\mathbf{r},k)\}$ where $\mathcal{D}$ is the inverse differential transform operator.

It is noted that the differential transform $U(\mathbf{r},h)$ is nothing but the coefficient of the Taylor expansion series of $u(\mathbf{x},t)$. Nevertheless, the DTM is different from to the classical Taylor expansion series method in determining the coefficients of the Taylor expansion series. In order to obtain the differential transform, the DTM provides a recursive equation which is derived by collecting the coefficients of the Taylor expansion series with the order of $(x_i - \tilde{x}_i)^{r_i}(t - \tilde{t})^k$ for all functions in the given differential equation. This results in a recursive equation corresponding to the given differential equation. As pointed out in [13], for nonlinear functions with several variables (even if simple) the corresponding differential transform would be very complicated due to multiple summations and it takes a lot of computational time. To see an example we refer the reader to [13]. A comprehensive study of the Taylor series expansion method and the DTM together with their differences and similarities can be found in [1, 4].

To overcome on the problem of the DTM, Jang in [13] proposed the PDTM to solve Eq. (2). In the PDTM, instead of considering the Taylor expansion series of $u(\mathbf{x}, t)$ for all variables $\mathbf{x}$ and t, the Taylor expansion series of $u(\mathbf{x}, t)$ is written with respect to some variable $x_p \in \{x_1, \dots, x_n, t\}$. Without loss of generality, we assume $x_p = t$. In this case, $u(\mathbf{x}, t)$ can be expressed as

$$u(\mathbf{x}, t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(\mathbf{x}, \tilde{t}) \right] (t - \tilde{t})^k. \tag{4}$$

Similar to the DTM, we define PDTM operator $\mathcal{P}_{\mathcal{D}}$ and inverse PDTM operator $\mathcal{P}_{\mathcal{D}}^{-1}$ as following

$$\begin{aligned} U(\mathbf{x}, k) &= \mathcal{P}_{\mathcal{D}}\{u(\mathbf{x}, t)\} = \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(\mathbf{x}, \tilde{t}) \right], \\ u(\mathbf{x}, t) &= \mathcal{P}_{\mathcal{D}}^{-1}\{U(\mathbf{x}, k)\}. \end{aligned}$$

In this case, we have

$$u(\mathbf{x}, t) = \sum_{k=0}^{\infty} U(\mathbf{x}, k) t^k.$$

For simplicity, throughout the paper we assume that $\tilde{t} = 0$. Here, we mention that one can use

$$u_n(\mathbf{x}, t) = \sum_{k=0}^{n} U(\mathbf{x}, k) t^k,$$

as an approximate solution to Eq. (2). The next theorem provides some fundamental operations performed by the projected differential transform.

Theorem 1. *Let* $U(\mathbf{x}, k) = \mathcal{P}_{\mathcal{D}}\{u(\mathbf{x}, t)\}$, $V(\mathbf{x}, k) = \mathcal{P}_{\mathcal{D}}\{v(\mathbf{x}, t)\}$ *and* $W(\mathbf{x}, k) = \mathcal{P}_{\mathcal{D}}\{w(\mathbf{x}, t)\}$, *where* $u(\mathbf{x}, t), v(\mathbf{x}, t)$ *and* $w(\mathbf{x}, t)$ are three analytic functions.

a) If $w(\mathbf{x}, t) = c_1 u(\mathbf{x}, t) \pm c_2 v(\mathbf{x}, t)$, *where* $c_1, c_2 \in \mathbb{R}$, *then* $W(\mathbf{x}, k) = c_1 U(\mathbf{x}, k) \pm c_2 V(\mathbf{x}, k)$.

b) If $w(\mathbf{x}, t) = \frac{d^m}{dt^m} u(\mathbf{x}, t)$, *then* $W(\mathbf{x}, k) = \frac{(k+m)!}{k!} U(\mathbf{x}, k+m)$.

c) If $w(\mathbf{x},t) = u(\mathbf{x},t)v(\mathbf{x},t)$*, then* $W(\mathbf{x},k) = \sum_{\ell=0}^{k} U(\mathbf{x},\ell)V(\mathbf{x},k-\ell)$.
d) If $w(\mathbf{x},t) = \frac{\partial^m}{\partial x_i^m} u(\mathbf{x},t)$*, then* $W(\mathbf{x},k) = \frac{\partial^m}{\partial x_i^m} U(\mathbf{x},t)$.

Proof. This theorem is a straightforward generalization of the single variable case of the DTM (see for example [2]) and its proof is omitted. □

In the sequel, we apply the PDTM to solve the CMKdV equation (1).

3. Application of PDTM to the CMKdV equation

Let $u(x,t) = r(x,t) + is(x,t)$. In this case, Eq. (1) is transformed into the coupled form

$$r_t + r_{xxx} + \beta((r^2+s^2)r)_x = 0, \tag{5}$$
$$s_t + s_{xxx} + \beta((r^2+s^2)s)_x = 0. \tag{6}$$

If we assume

$$R(x,k) = \mathcal{P}_{\mathcal{D}}\{r(x,t)\} = \frac{1}{k!}\frac{\partial^k r(x,t)}{\partial t^k}\bigg|_{t=0},$$

and

$$S(x,k) = \mathcal{P}_{\mathcal{D}}\{s(x,t)\} = \frac{1}{k!}\frac{\partial^k s(x,t)}{\partial t^k}\bigg|_{t=0},$$

then

$$r(x,t) = \sum_{k=0}^{\infty} R(x,k)t^k, \qquad s(x,t) = \sum_{k=0}^{\infty} S(x,k)t^k. \tag{7}$$

Now, we first apply the projected differential transform to Eq. (5). From Theorem 1, we have

$$\mathcal{P}_{\mathcal{D}}\{r_t(x,t)\} + \mathcal{P}_{\mathcal{D}}\{r_{xxx}(x,t)\} + \beta\mathcal{P}_{\mathcal{D}}\{((r(x,t)^2 + s(x,t)^2)r(x,t))_x\} = 0,$$

which is equivalent to

$$\begin{aligned}&(k+1)R(x,k+1) + \frac{\partial^3}{\partial x^3}R(x,k)\\ &\quad +\beta\frac{\partial}{\partial x}\sum_{m=0}^{k}\sum_{i=0}^{k-m}[R(x,m)R(x,i) \quad + \quad S(x,m)S(x,i)]R(x,k-m-i) = 0.\end{aligned}$$

Therefore, we have

$$\begin{aligned}R(x,k+1) &= \frac{1}{k+1}\{-\frac{\partial^3}{\partial x^3}R(x,k)\\ &\quad -\beta\frac{\partial}{\partial x}\sum_{m=0}^{k}\sum_{i=0}^{k-m}[R(x,m)R(x,i) + S(x,m)S(x,i)]R(x,k-m-i)\}.\end{aligned} \tag{8}$$

In the same way, we obtain

$$\begin{aligned} S(x,k+1) &= \frac{1}{k+1}\{-\frac{\partial^3}{\partial x^3}S(x,k) \\ &-\beta\frac{\partial}{\partial x}\sum_{m=0}^{k}\sum_{i=0}^{k-m}[R(x,m)R(x,i)+S(x,m)S(x,i)]S(x,k-m-i)\}. \end{aligned} \qquad (9)$$

Now, let $u(x,0)=u_0(x)=r_0(x)+is_0(x)$ be the initial condition. In this case, we have $r(x,0)=r_0(x)$ and $s(x,0)=s_0(x)$. Therefore, $R(x,0)=r_0(x)$ and $S(x,0)=s_0(x)$. Hence, by Eqs. (8) and (9) one can compute $R(x,k)$ and $S(x,k)$ for all $k\geq 1$. Then, $r(x,t)$ and $s(x,t)$ are computed via (7).

In the next section we present three illustrative examples.

4. Illustrative examples

In this section we compare the numerical results of the PDTM with those of the DTM and the VIM. To do so, we first give a brief description of the application of these methods to the CMKdV problem.

Let $r(x,t)$ be the real part of $u(x,t)$. Then, by invoking Eq. (3), we have

$$r(x,t)=\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}R(h,k)(x-\tilde{x})^h(t-\tilde{t})^k, \qquad (10)$$

where

$$R(h,k)=\frac{1}{h!k!}\frac{\partial^{h+k}r(x,t)}{\partial x^h\partial t^k}\bigg|_{(\tilde{x},\tilde{t})},$$

is the two-dimensional differential transform of $r(x,t)$. Similarly, let $S(h,k)$ be the two-dimensional differential transform of $s(x,t)$. We can state a theorem like Theorem 1 to present fundamental properties of the two-dimensional differential transform but, here, we refer the reader to [18]. Applying differential transform $\mathcal{D}$ to both sides of Eqs. (5) and (6) gives

$$\begin{aligned} R(h,k+1) &= \frac{-1}{k+1}[(h+3)(h+2)(h+1)R(h+3,k) \\ &+\beta(h+1)\sum_{r=0}^{h+1}\sum_{\ell=0}^{h+1-r}\sum_{s=0}^{k}\sum_{p=0}^{k-s}\left(R(r,k-s-p)R(\ell,s)+S(r,k-s-p)S(\ell,s)\right)R(h+1-r-\ell,p)], \end{aligned}$$

and

$$\begin{aligned} S(h,k+1) &= \frac{-1}{k+1}[(h+3)(h+2)(h+1)S(h+3,k) \\ &+\beta(h+1)\sum_{r=0}^{h+1}\sum_{\ell=0}^{h+1-r}\sum_{s=0}^{k}\sum_{p=0}^{k-s}\left(R(r,k-s-p)R(\ell,s)+S(r,k-s-p)S(\ell,s)\right)R(h+1-r-\ell,p)]. \end{aligned}$$

By using the Taylor expansion series of initial condition $r(x,0)$ and $s(x,0)$ we can compute $R(i,0)$ and $S(i,0)$ for $i \geq 0$. Then, $U(h,k)$'s can be computed by the above recursive formulas. In real applications, for two given natural numbers p and q, the function $r(x,t)$ is approximated by a truncated version of Eq. (10) of the form

$$r(x,t) \approx \sum_{h=0}^{p} \sum_{k=0}^{q} R(h,k)(x-\tilde{x})^h (t-\tilde{t})^k.$$

Similarly, the function $s(x,t)$ is approximated by a truncated version of its Taylor expansion series.

We now investigate the application of the VIM to the CMKdV equation. According to the variational method [8, 9, 10], we can write down the following correction functionals

$$\begin{aligned} r_{n+1}(x,t) &= r_n(x,t) + \int_0^t \lambda_1 \left(\frac{\partial r_n(x,\tau)}{\partial \tau} + \frac{\partial^3 \tilde{r}_n(x,\tau)}{\partial x^3} + \beta \frac{\partial}{\partial x} \left((\tilde{r}(x,\tau)^2 + \tilde{s}(x,\tau)^2) \tilde{r}_n(x,\tau) \right) \right) d\tau, \\ s_{n+1}(x,t) &= s_n(x,t) + \int_0^t \lambda_2 \left(\frac{\partial s_n(x,\tau)}{\partial \tau} + \frac{\partial^3 \tilde{s}_n(x,\tau)}{\partial x^3} + \beta \frac{\partial}{\partial x} \left((\tilde{r}(x,\tau)^2 + \tilde{s}(x,\tau)^2) \tilde{s}_n(x,\tau) \right) \right) d\tau, \end{aligned}$$

where λ_1 and λ_2 are general Lagrange multipliers, and can be identified optimally by the variational theory, $r_0(x,t)$ and $s_0(x,t)$ are initial approximations, $\tilde{r}_n(x,t)$ and $\tilde{s}_n(x,t)$ are considered as restricted variations, i.e. $\delta \tilde{r}_n = 0$ and $\delta \tilde{s}_n = 0$. It is easy to see that $\lambda_1 = \lambda_2 = -1$ (see [8] for more details). In this case the VIM for solving the CMKdV equation takes the following form

$$\begin{aligned} r_{n+1}(x,t) &= r_n(x,t) - \int_0^t \left(\frac{\partial r_n(x,\tau)}{\partial \tau} + \frac{\partial^3 r_n(x,\tau)}{\partial x^3} + \beta \frac{\partial}{\partial x} \left((r(x,\tau)^2 + s(x,\tau)^2) r_n(x,\tau) \right) \right) d\tau, \\ s_{n+1}(x,t) &= s_n(x,t) - \int_0^t \left(\frac{\partial s_n(x,\tau)}{\partial \tau} + \frac{\partial^3 s_n(x,\tau)}{\partial x^3} + \beta \frac{\partial}{\partial x} \left((r(x,\tau)^2 + s(x,\tau)^2) s_n(x,\tau) \right) \right) d\tau, \end{aligned}$$

where $r_0(x,t) = r(x,0)$ and $s_0(x,t) = s(x,0)$.

We now give the numerical results. For the sake of comparison, we have selected three examples where the exact solution already exists, which will ultimately show the simplicity, effectiveness and exactness of the proposed method. All the computations are performed in MAPLE software on a Pentium 4 PC, with a 3.06 GHz CPU and 1.00GB of RAM.

Example 1. A solitary wave solution for the CMKdV equation can be described as (see [15])

$$u(x,t) = \sqrt{\frac{2c}{\beta}} \operatorname{sech}(\sqrt{c}(x - x_0 - ct)) \exp(i\theta_0). \tag{11}$$

This single solitary wave of amplitude $\sqrt{2c/\beta}$ is centered at x_0 and moving to the right with velocity c. In this example, the value of parameters are chosen as $\beta = 2$, $c = 1$, $x_0 = 15$ and $\theta_0 = 0$. The initial condition is given by $u(x,0) = \operatorname{sech}(x - 15)$. It is clear that for a y-polarized ($\theta_0 = 0$) solitary wave solution of the CMKdV equation, the imaginary part is zero. Therefore, using Eqs. (8) and (9), the proposed method to compute $R(x,i)$, $i = 0,1,\ldots$, takes the following form:

$$\begin{aligned} R(x,k+1) &= \frac{1}{k+1}\{-\frac{\partial^3}{\partial x^3}R(x,k) \\ &-\beta\frac{\partial}{\partial x}\sum_{m=0}^{k}\sum_{i=0}^{k-m}R(x,m)R(x,i)R(x,k-m-i)\}, \quad k = 0,1,\ldots. \end{aligned}$$

Starting with $R(x,0) = \operatorname{sech}(x-15)$, we get

$$\begin{aligned} R(x,0) &= \operatorname{sech}(x-15), \\ R(x,1) &= \frac{\sinh(x-15)}{\cosh^2(x-15)}, \\ R(x,2) &= \frac{1}{2}\frac{\cosh^2(x-15)-2}{\cosh^3(x-15)}, \\ R(x,3) &= \frac{1}{6}\frac{\sinh(x-15)\left(\cosh^2(x-15)-6\right)}{\cosh^4(x-15)}, \\ R(x,4) &= \frac{1}{24}\frac{\cosh^4(x-15)-20\cosh^2(x-15)+24}{\cosh^5(x-15)}, \\ R(x,5) &= \frac{1}{120}\frac{\sinh(x-15)\left(\cosh^4(x-15)-60\cosh^2(x-15)+120\right)}{\cosh^6(x-15)}, \\ &\vdots \end{aligned}$$

An analytical solution have been obtained by substituting $R(x,i)$, $i = 0,1,\ldots$, in Eq. (7). For the comparison purpose between the exact solution and the computed solution by the PDTM, the multivariate Taylor expansion series of $r(x,t)$ is applied at $(x,t) = (15,0)$. Hence, we get

$$\begin{aligned} r(x,t) &= 1-\frac{1}{2}(x-15)^2+t(x-15)-\frac{1}{2}t^2+\frac{5}{24}(x-15)^4-\frac{5}{6}t(x-15)^3 \\ &+\frac{5}{4}t^2(x-15)^2-\frac{5}{6}t^3(x-15)+\frac{5}{24}t^4+\ldots, \end{aligned}$$

which is exactly the same as the Taylor expansion series of the exact solution. In Figure 1, the modulus of the approximate solution $u_5(x,t)$ and the exact solutions of the CMKdV equation with initial condition (11) for $\beta = 2$, $x_0 = 15$, $\theta_0 = 0$ and velocity $c = 1$ are presented.

To see the effectiveness of the PDTM we compare its numerical results with those of the DTM and the VIM. To do so, the numerical results including $|u|$ and the CPU times (in seconds) to compute the approximate solution by these three methods at $(0.5,0.5)$, $(10,0.5)$ and $(20,0.5)$ are reported in Table 1. We mention that, for a given

Table 1: Numerical comparison between the VIM, the DTM and the PDTM for Example 1.

	Exact	VIM, $n=4$	DTM, $n=5$	PDTM, $n=5$
$(0.5, 0.5)$	6.118046410e-7	6.120469032e-7	6.118046410e-7	6.117842212e-7
$(10, 0.5)$	8.173406367e-3	8.159142736e-3	7.235910676e-4	8.173141249e-3
$(20, 0.5)$	2.221525150e-2	2.223279605e-2	1.894569839e-2	2.221494954e-2
CPU times	-	12.969	102.344	1.391

natural number n, in the DTM a finite series of degree n with respect to t and in the PDTM a polynomial of degree n is computed. Here, we mention that we could not compute any solution by the VIM for $n = 5$ in a reasonable time. As we see the PDTM in general is superior to the other methods. Another observation which can be posed here is that the DTM have not provided a good solution for points which are far from $(0, 0)$, whereas the approximate solutions given by the PDTM are in good agreement with the exact solution.

Example 2. In this example, consider Eq. (11) with $\beta = 2$, $c = 1$, $x_0 = 15$ and $\theta_0 = \pi/4$. In this case, the initial conditions are given by

$$r(x,0) = \frac{\sqrt{2}}{2}\,\operatorname{sech}(x-15) = s(x,0).$$

Obviously, the real and the imaginary parts of the exact solutions are the same. Therefore, we present only the real part of the computed solution by using $R(x,i)$, $i = 0, 1, \ldots,$ as following:

$$\begin{aligned}
R(x,0) &= \frac{\sqrt{2}}{2}\operatorname{sech}(x-15),\\
R(x,1) &= \frac{\sqrt{2}}{2}\frac{\sinh(x-15)}{\cosh^2(x-15)},\\
R(x,2) &= \frac{\sqrt{2}}{4}\frac{\cosh^2(x-15)-2}{\cosh^3(x-15)},\\
R(x,3) &= \frac{\sqrt{2}}{12}\frac{\sinh(x-15)\left(\cosh^2(x-15)-6\right)}{\cosh^4(x-15)},\\
R(x,4) &= \frac{\sqrt{2}}{48}\frac{\cosh^4(x-15)-20\cosh^2(x-15)+24}{\cosh^5(x-15)},\\
R(x,5) &= \frac{\sqrt{2}}{240}\frac{\sinh(x-15)\left(\cosh^4(x-15)-60\cosh^2(x-15)+120\right)}{\cosh^6(x-15)},\\
&\vdots
\end{aligned}$$

Now, $r(x,t)$ is computed via (7). Similar to the previous example, the multivariate Taylor expansion series has been applied to compare the computed solution by the

Table 2: Numerical comparison between the VIM, the DTM and the PDTM for Example 2.

	Exact	VIM, $n=2$	DTM, $n=4$	PDTM, $n=4$
$(0.5, 0.5)$	6.118046410e-7	6.304345780e-7	6.118046410e-7	6.120469031e-7
$(10, 0.5)$	8.173406367e-3	8.416774834e-3	3.290887604e-4	8.176612266e-3
$(20, 0.5)$	2.221525150e-2	2.190311256e-2	4.570806811e-3	2.221147851e-2
CPU times	-	5.641	54.016	7.625

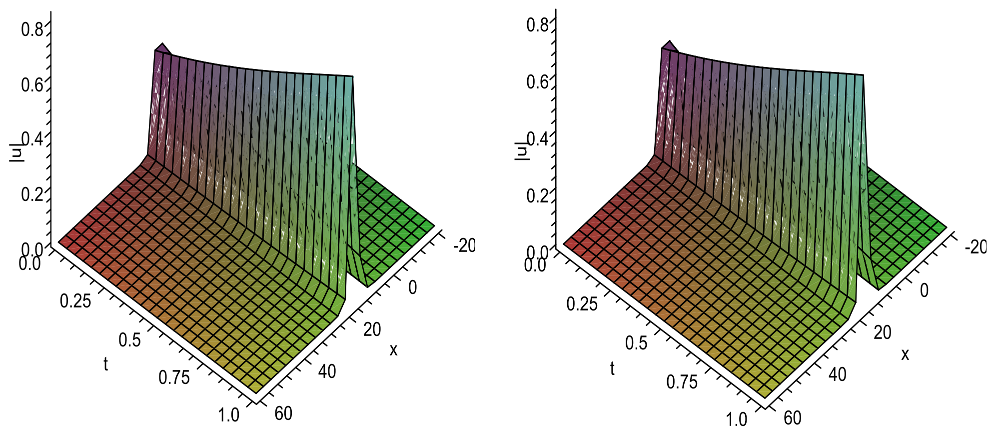

Figure 1: Comparison between the approximate solution (left) and the exact solution (right) for Example 1

PDTM and the exact solution. Taylor expansion series of the computed solution for real part at $(x, 0) = (15, 0)$ is given by

$$\begin{aligned} r(x,t) = {} & \frac{1}{2}\sqrt{2} - \frac{1}{4}\sqrt{2}\,(x-15)^2 + \frac{1}{2}\sqrt{2}t\,(x-15) - \frac{1}{4}\sqrt{2}t^2 + \frac{5}{48}\sqrt{2}\,(x-15)^4 \\ & - \frac{5}{12}\sqrt{2}t\,(x-15)^3 + \frac{5}{8}\sqrt{2}t^2\,(x-15)^2 - \frac{5}{12}\sqrt{2}t^3\,(x-15) + \frac{5}{48}t^4\sqrt{2} \\ & + \ldots, \end{aligned}$$

which is the Taylor expansion series of the exact solution. For more investigation, the modulus of the approximate solution $u_5(x,t) = r_5(x,t) + is_5(x,t)$ and the exact solution with $\beta = 2$, $x_0 = 15$, $\theta_0 = \pi/4$ and velocity $c = 1$ are shown in Figure 2.

Similar to the previous example we compare the numerical results of the PDTM, the DTM and the VIM in Table 2. This table shows the superiority of the PDTM method over other two methods. It is mentioned that we could not obtain any solution by the VIM for $n = 3$ in a reasonable time.

Example 3. As the final example, consider interaction of two solitary waves, with

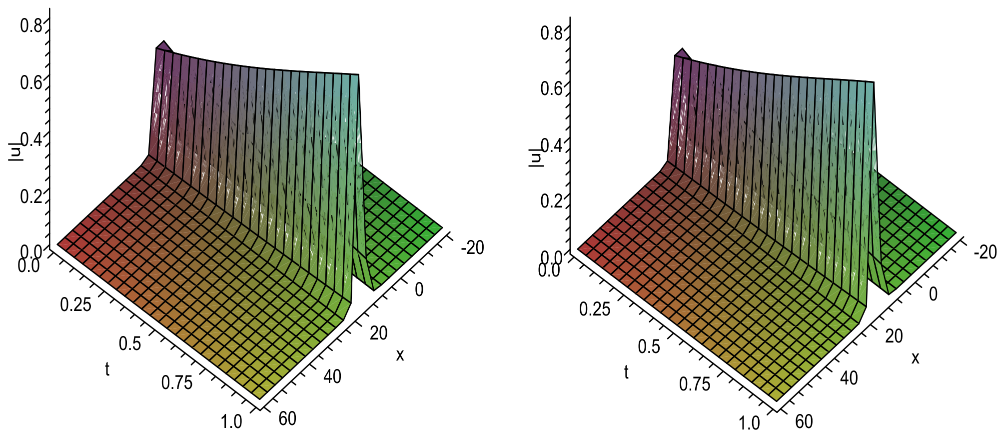

Figure 2: Comparison between the approximate solution (left) and the exact solution (right) for Example 2

the initial condition of the form

$$u(x,0) = \sqrt{\frac{2c_1}{\beta}}\ \operatorname{sech}[\sqrt{c_1}(x-25)]\exp(i\theta_1) + \sqrt{\frac{2c_2}{\beta}}\ \operatorname{sech}[\sqrt{c_2}(x-48)]\exp(i\theta_2), \quad (12)$$

where the parameters are chosen as $c_1 = 2$, $c_2 = 1/2$, $x_1 = 25$, $x_2 = 48$, $\theta_1 = 0$, $\theta_2 = \pi/2$ and $\beta = 2$. We use the PDTM to compute the solution of the CMKdV equation with the initial condition (12). Since, the expressions for $R(x,i)$ and $S(x,i)$ are too complicated, the presentation of them are omitted here. Instead, we give the Taylor expansion series of the computed solutions $r(x,t)$ and $s(x,t)$ at $(x_0,t_0) = (25,0)$ and $(x_0,t_0) = (48,0)$, respectively, as following

$$\begin{aligned} r(x,t) &= 1.414 - 1.414\,(x-25)^2 + 5.657t\,(x-25) - 5.657t^2 + 1.179\,(x-25)^4 \\ &\quad -9.429t\,(x-25)^3 + 28.284t^2\,(x-25)^2 - 37.712t^3\,(x-25) \\ &\quad +18.856t^4 + \ldots, \end{aligned}$$

$$\begin{aligned} s(x,t) &= 0.707 - 0.177\,(x-48)^2 + 0.177t\,(x-48) - 0.044t^2 + 0.037\,(x-48)^4 \\ &\quad -0.074t\,(x-48)^3 + 0.056t^2\,(x-48)^2 - 0.018t^3\,(x-48) + 0.002t^4 \\ &\quad + \ldots. \end{aligned}$$

It is easy to verify that the obtained expansions are the same as the Taylor expansion series of real and imaginary part of the exact solution. For more investigation, the simulation of the approximate solution $u_6(x,t) = r_6(x,t) + is_6(x,t)$ and the exact solution for $0 \leq t \leq 0.5$ are plotted in Figure 3.

Table 3 presents a comparison between the VIM, the DTM and the PDTM. As we observe the PDTM is the best among the investigated methods. It is noted that

Table 3: Numerical comparison between the VIM, the DTM and the PDTM for Example 3.

	Exact	VIM, $n=2$	DTM, $n=4$	PDTM, $n=4$
$(0.5, 0.5)$	3.129223549e-15	3.411365694e-15	3.129544135e-15	3.149580817e-15
$(10, 0.5)$	4.212990947e-10	1.015099166e-9	1.970248025e-12	4.870293015e-10
$(20, 0.5)$	5.840285764e-4	1.407122767e-3	3.215762446e-11	6.751253528e-4
CPU times	-	68.562	54.016	10.703

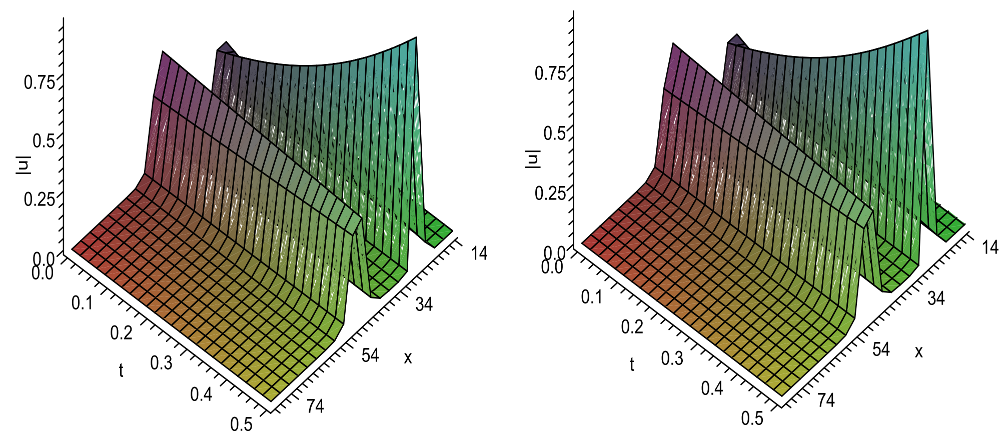

Figure 3: Comparison between the approximate solution (left) and the exact solution (right) for Example 3

the VIM for $n = 3$ and the DTM for $n = 5$ have not provided any solution in a reasonable time.

5. Conclusion

In this paper, the PDTM has been successfully applied to obtain an approximate solution of the CMKdV equation. We have presented three illustrate examples which show the efficiency of the method. Moreover, numerical experiments of the PDTM have been compared with those of the VIM and the DTM. Numerical results show that the PDTM is the best among these methods. It may be concluded that the reduction in the size of computational domain and the reliability of the method give a wide applicability to solve various kinds of linear or nonlinear problems. We have found that in general the PDTM is easier and more efficient than DTM method.

Acknowledgments

The authors are gratefully indebted to the anonymous referees and professor Melvin Scott for their valuable suggestions and remarks which substantially improved the quality of this paper.

References

[1] S. Abbasbandy and C. Bervillier, *Analytic continuation of Taylor series and the boundary value problems of some nonlinear ordinary differential equations*, Applied Mathematics and Computation **218** (2011) 2178-2199.

[2] I. H. Abdel-Halim Hassan, *Application to differential transformation method for solving systems of differential equations*, Applied Mathematical Modelling **32** (2008) 2552-2559.

[3] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer, Boston, 1994.

[4] C. Bervillier, *Status of the differential transformation method*, Applied Mathematics and Computation **218** (2012) 10158-10170.

[5] L. Debnath, *Nonlinear partial differential equations for scientist and engineers*, Birkhauser, Boston, MA, 1997.

[6] H. A. Erbay, *Nonlinear transverse waves in a generalized elastic solid and the complex modified Kortewegde Vries equation*, Phys. Scripta **58** (1998) 9-14.

[7] O. B. Gorbacheva, L. A. Ostrovsky, *Nonlinear vector waves in a mechanical model of a molecular chain*, Physica D **8** (1983) 223-228.

[8] J. H. He, *Approximate analytical solution for seepage flow with fractional derivatives in porous media*, Comput. Methods Appl. Mech. Engg. **167** (1998) 57-68.

[9] J. H. He, *Variational iteration method - a kind of non-linear analytical technique: some examples*, International Journal of Non-Linear Mechanics **34** (1999) 699-708.

[10] J. H. He and X. H. Wu, *Variational iteration method: New development and applications*, Computers and Mathematics with Applications **54** (2007) 881-894.

[11] M. S. Ismail, *Numerical solution of complex modified Korteweg-de Vries equation by collocation method*, Communications in Nonlinear Science and Numerical Simulation **14** (2009) 749-759.

[12] M. S. Ismail, *Numerical solution of complex modified Kortewegde Vries equation by PetrovGalerkin method*, Applied Mathematics and Computation, Applied Mathematics and Computation **202** (2008) 520-531.

[13] B. Jang, *Solving linear and nonlinear initial value problems by the projected differential transform method*, Computer Physics Communications, Computer Physics Communications **181** (2010) 848-854.

[14] C. F. F. Karney, A. Sen, F. Y. F. Chu, *Nonlinear evolution of lower hybrid waves*, Phys. Fluids **22** (1979) 940-952.

[15] A. Korkmaz and I. Dag, *Solitary wave simulations of Complex Modified Kortewegde Vries Equation using differential quadrature method*, Computer Physics Communications **180** (2009) 1516-1523.

[16] G. M. Muslu and H. A. Erbay, *A Split-Step Fourier Method for the Complex Modified Korteweg-de Vries Equation*, Computers and Mathematics with Applications, **45** (2003) 503-514.

[17] H. Roohani and D. K. Salkuyeh, *New general solutions for the general elliptic and auxiliary equations and application to the coupled KdV equation*, International Journal of Computer Mathematics **87** (2010) 2760-2768.

[18] A. Tari, M.Y. Rahimi, S. Shahmorad and F. Talati, *Solving a class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method*, Journal of Computational and Applied Mathematics **228** (2009) 70-76.

[19] M. Wadati, *Introduction to solitons*, Pramana. J. Phys **57** (2001) 841-847.

[20] M. Wadati, *The exact solution of the Kortweg-de Vries equation*, J. Phys Soc Jpn, **32** (1972) 1681-1687.

[21] A. M. Wazwaz, P*artial differential equations: methods and appplications*, Balkema, Rotterdam, 2002.

[22] J. K. Zhou, *Differential Transformation and its Application for Electrical Circuits*, Huazhong University Press, China, 1986 (in Chinese).