New General Solutions for the General Elliptic and Auxiliary Equations and Application to the Coupled KdV Equation

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Abstract: In this paper, we first obtain generalized soliton solutions of the general elliptic and auxiliary equations by the Exp-function method. Then by the obtained solutions, we find new and more general solutions of the coupled KdV equation.

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1. Introduction

The nonlinear phenomena play a crucial rule in a variety of scientific fields such as fluid mechanics, optical fibers, solid state physics, chemical kinetics and geochemistry and are modeled by nonlinear partial differential equations (NPDEs) [4, 19, 21, 22]. Searching and constructing exact solutions of NPDEs is one of the most important tasks in soliton theory [4, 22], because, exact solutions help one to well understand the mechanism of the complicated physical phenomena and dynamical processes modeled by these NPDEs.

With the development of soliton theory, many powerful methods for obtaining exact solutions of NPDEs have been developed, such as Bäcklund transformation method [15], sine-cosine method [23, 24, 25], homogeneous balance method [20], homotopy perturbation method [6, 9, 17], tanh-function method [27, 28], F-expansion method [29, 30, 36], Jacobi elliptic function expansion method [14, 7], Painlevé method [2, 3], and so on. As we know, the Painlevé test is a way to test the integrability of NPDEs, but by exploiting the information provided by the Painlevé test one can obtain exact solutions of NPDEs. In fact, by using the Painlevé expansion method one can obtain the exact solution of NPDEs (see for example [13]).

Recently, He and Wu [10], proposed the Exp-function method to obtain exact solutions of NPDEs. The Exp-function method has been successfully applied to many kinds of NPDEs (see for example [1, 8, 10, 11, 16] and references therein). In the Exp-function method, as we shall see in the section 2, the NPDE is transformed into an ordinary differential equation and the Exp-function is used to solve it. Whereas, in the Painlevé method, which is similar to the Exp-function method, the NPDEs are solved directly and the method is slightly complicated. Moreover, the Exp-function method leads to not only generalized soliton solutions but also periodic solutions [31]. We mention that the solution procedure of the Exp-function method is simply implemented on using a computer software package, such as Matlab or Mathematica.

In most of these methods, the exact solutions of NPDEs are written as a polynomial in several elementary or special functions which satisfy a first-order ordinary differential equation called sub-equation, for example, elliptic equation, Riccati equation and auxiliary equation [26, 30, 31]. Obviously, more solutions of these sub-equations result in more exact solutions of the considered NPDEs.

In this paper, we consider the general elliptic equation [26, 33, 34]

$$(\frac{d\phi(\xi)}{d\xi})^2 = h_0 + h_1\phi(\xi) + h_2\phi^2(\xi) + h_3\phi^3(\xi) + h_4\phi^4(\xi). \tag{1}$$

When $h_0 = h_1 = 0$, the general elliptic equation is reduced to the auxiliary ordinary equation

$$(\frac{d\phi(\xi)}{d\xi})^2 = h_2\phi^2(\xi) + h_3\phi^3(\xi) + h_4\phi^4(\xi). \tag{2}$$

We use the Exp-function method to seek new generalized soliton solutions of the general elliptic and the auxiliary equations. Then, we show that by choosing different parameters these solutions result in several available solutions found in the literature. Finally, the obtained solutions are used to construct new exact solutions of the coupled KdV equation [12]

$$\begin{cases} u_t - 6uu_x - 2vv_x - u_{xxx} &= 0, \\ v_t + 3uv_x + v_{xxx} &= 0, \end{cases} \tag{3}$$

where $u = u(x,t)$ and $v = v(x,t)$ are unknown functions.

2. Exp-function method for general elliptic and auxiliary equations

2.1 Exp-function method for general elliptic equation

Let η be a complex variable defined by

$$\eta = k\xi + \omega,$$

where k is a constant to be determined later and ω is an arbitrary constant. This transformation changes Eq. (1) into

$$k^2 \ \phi'^{\ 2} = h_0 + h_1\phi + h_2\phi^2 + h_3\phi^3 + h_4\phi^4, \tag{4}$$

where prime denotes the derivative with respect to η and $\phi = \phi(\eta)$. In some special cases, when $h_i \neq 0, \ i = 0, 1, 2, 3, 4$, it may exist three parameters r, p and q such that

$$k^2 \ \phi'^{\ 2} = h_0 + h_1\phi + h_2\phi^2 + h_3\phi^3 + h_4\phi^4 = (r + p\phi + q\phi^2)^2. \tag{5}$$

Eq. (5) is satisfied only if

$$h_0 = r^2, \quad h_1 = 2rp, \quad h_2 = 2rq + p^2, \quad h_3 = 2pq, \quad h_4 = q^2. \tag{6}$$

In this case, Eq. (5) can be rewritten as

$$k\phi' = r + p\phi + q\phi^2. \tag{7}$$

According to Exp-function method, we assume that the solution of Eq. (5) can be expressed in the following form

$$\phi(\eta) = \frac{\sum_{n=-d}^{c} a_n \exp(n\eta)}{\sum_{n=-f}^{e} b_n \exp(n\eta)}, \tag{8}$$

where c, d, e and f are positive integers which are unknown to be determined, and a_n's, b_n's are unknown constants. In order to determine values of c and e, we balance the linear term of highest order in Eq. (7), with the highest order nonlinear term [11]. By a little computation we see that

$$\phi' = \frac{A_1 \exp((c+e)\eta) + ...}{A_2 \exp(2e\eta) + ...}, \tag{9}$$

and

$$\phi^2 = \frac{A_3 \exp(2c\eta) + ...}{A_4 \exp(2e\eta) + ...}, \tag{10}$$

where A_i's are determined coefficients written as such only for simplicity. Balancing highest order of Exp-function in Eqs. (9) and (10), we get

$$c + e = 2c,$$

which results in $e = c$. Similarly to determine values of d and f, we balance the linear term of lowest order in Eq. (7). It is easy to see that

$$\phi' = \frac{... + B_1 \exp(-(d+f)\eta)}{... + B_2 \exp(-2f\eta)}, \tag{11}$$

and

$$\phi^2 = \frac{... + B_3 \exp(-2d\eta)}{... + B_4 \exp(-2e\eta)}, \tag{12}$$

where B_i's are determined coefficients written as such only for simplicity. Balancing lowest order of Exp-function in Eqs. (11) and (12), we have

$$-(d + f) = -2d,$$

which leads to the result $f = d$. For simplicity, let $e = c = 1$ and $f = d = 1$. Then Eq. (8) becomes

$$\phi(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{13}$$

Substituting Eq. (13) into Eq. (5), we have

$$\begin{aligned} &\tfrac{1}{A}[C_1 \exp(4\eta) + C_2 \exp(3\eta) + C_3 \exp(2\eta) + C_4 \exp(\eta) + C_5 \\ &+C_6 \exp(-\eta) + C_7 \exp(-2\eta) + C_8 \exp(-3\eta) + C_9 \exp(-4\eta)] = 0. \end{aligned} \tag{14}$$

Equating the coefficients of $\exp(i\eta)$, $i = 4, 3, \ldots, -3, -4$, to zero, we obtain $C_i = 0$, $i = 1, 2, \ldots, 9$. Solving this system of algebraic equations with the aid of Maple, we conclude that

$$a_0 = 0,\ b_0 = 0,\ k = \frac{1}{2}\sqrt{p^2 - 4qr},\ b_1 = \frac{p - \sqrt{p^2 - 4qr}}{-2r} a_1,\ b_{-1} = \frac{p + \sqrt{p^2 - 4qr}}{-2r} a_{-1}, \tag{15}$$

where a_1 and a_{-1} are free parameters. Substituting Eqs. (15) into Eq. (13), we obtain the following general solution of Eq. (1)

$$\phi(\xi) = \frac{a_1 \exp(k\xi + \omega) + a_{-1} \exp(-k\xi - \omega)}{\frac{p-\sqrt{p^2-4qr}}{-2r} a_1 \exp(k\xi + \omega) + \frac{p+\sqrt{p^2-4qr}}{-2r} a_{-1} \exp(-k\xi - \omega)}. \tag{16}$$

Many solutions of the general elliptic equation given in several papers can be obtained by setting different values for the free parameters in Eq. (16). For example, since ω is an arbitrary constant, we assume $\omega = 0$. Then, we consider two cases $p^2 - 4qr > 0$ and $p^2 - 4qr < 0$.

In the first case, if we set $a_1 = p + 2k$, and $a_{-1} = p - 2k$, then we can obtain

$$\phi_1(\xi) = -\frac{1}{2q}[p + \sqrt{p^2 - 4rq}\tanh(\frac{\sqrt{p^2 - 4qr}}{2}\xi)],$$

which is the function ϕ_1^I in [26]. For $a_1 = p + 2k$, and $a_{-1} = \pm(-p + 2k)i$, Eq. (16) results in

$$\phi_2(\xi) = -\frac{1}{2q}[p + \sqrt{p^2 - 4qr}[\tanh(\sqrt{p^2 - 4qr}\xi) \pm i\ \mathrm{sech}(\sqrt{p^2 - 4qr}\xi)]],$$

which is equal to ϕ_3^I in [26]. If we set $a_1 = a_{-1} = r$, then Eq. (16) becomes

$$\phi_3(\xi) = \frac{2r\cosh(\frac{\sqrt{p^2-4qr}}{2}\xi)}{\sqrt{p^2 - 4qr}\sinh(\frac{\sqrt{p^2-4qr}}{2}\xi) - p\cosh(\frac{\sqrt{p^2-4qr}}{2}\xi)},$$

which is the same solution as ϕ_8^I in [26].

Now we assume $p^2 - 4qr < 0$. If we set $a_1 = -p - 2k$, and $a_{-1} = -p + 2k$, then Eq. (16) becomes

$$\phi_4(\xi) = \frac{1}{2q}[-p + \sqrt{4rq - p^2}\tan(\frac{\sqrt{4qr - p^2}}{2}\xi)],$$

which is the same solution as ϕ_{13}^I in [26]. Assuming $a_1 = \pm(p + 2k)i$, and $a_{-1} = (-p + 2k)$, the Eq. (16) gives

$$\phi_5(\xi) = \frac{1}{2q}[-p + \sqrt{4qr - p^2}[\tan(\sqrt{4qr - p^2}\xi) \pm \sec(\sqrt{4qr - p^2}\xi)]].$$

This is the same solution as ϕ_{15}^I [26]. If we set $a_1 = a_{-1} = -r$, then Eq. (16) becomes

$$\phi_6(\xi) = -\frac{2r\cos(\frac{\sqrt{4qr-p^2}}{2}\xi)}{\sqrt{4qr - p^2}\sin(\frac{\sqrt{4qr-p^2}}{2}\xi) + p\cos(\frac{\sqrt{4qr-p^2}}{2}\xi)}.$$

which is the function ϕ_{20}^I in [26].

By choosing different values for the free parameters a_{-1} and a_1 we have obtained many solutions reported in [26] which we omit in the present discussion. Hence Eq. (16) is indeed a general solution of general elliptic equation.

2.2 Exp-function method for auxiliary equation

In this subsection, we use the Exp-function method to seek a general solution of the auxiliary equation

$$(\frac{d\phi(\xi)}{d\xi})^2 = A\phi^2(\xi) + B\phi^3(\xi) + C\phi^4(\xi). \tag{17}$$

Similar to the previous subsection, it is easy to see that the solution of Eq. (17) can be expressed in the following form

$$\phi(\eta) = \frac{a'_1 \exp(\eta) + a'_0 + a'_{-1} \exp(-\eta)}{b'_1 \exp(\eta) + b'_0 + b'_{-1} \exp(-\eta)}, \tag{18}$$

where $\eta = k\xi + \omega$ is a complex variable. Substituting Eq. (18) into Eq. (17), we obtain

$$\frac{1}{Q}[D_1 \exp(4\eta) + D_2 \exp(3\eta) + D_3 \exp(2\eta) + D_4 \exp(\eta) + D_5 \\ +D_6 exp(-\eta) + D_7 \exp(-2\eta) + D_8 \exp(-3\eta) + D_9 \exp(-4\eta)] = 0.$$

Equating the coefficients of $\exp(i\eta)$, $i = 4, 3, \ldots, -3, -4$, to zero, we have $D_i = 0$, $i = 1, \ldots, 9$. By solving this system of algebraic equations with the aid of Maple, we conclude: If $B \neq 0$, then

$$a'_1 = 0,\ a'_0 = -\frac{2Ab'_0}{B},\ a'_{-1} = 0,\ b'_{-1} = -\frac{1}{4}\frac{b'^2_0(4AC - B^2)}{b'_1 B^2},\ k = \varepsilon\sqrt{A},\ \varepsilon = \pm 1, \tag{19}$$

where b'_0 and $b'_1 \neq 0$ are free parameters; If $B = 0$ and $A \neq 0$, then

$$a'_1 = 0,\ a'_{-1} = 0,\ b'_{-1} = -\frac{1}{4}\frac{Ca'^2_0}{Ab'_1},\ b'_0 = 0,\ k = \varepsilon\sqrt{A},\ \varepsilon = \pm 1, \tag{20}$$

where a'_0 and $b'_1 \neq 0$ are free parameters; If $C = 0$ and $B \neq 0$, then

$$a'_1 = 0,\ a'_{-1} = 0,\ b'_{-1} = \frac{b'^2_0}{4b'_1},\ a'_0 = \frac{-2Ab'_0}{B},\ k = \varepsilon\sqrt{A},\ \varepsilon = \pm 1, \tag{21}$$

where b'_0 and $b'_1 \neq 0$ are free parameters.

Substituting (19) into Eq. (18), we obtain the following general solution of Eq. (17)

$$\phi(\xi)=\frac{-\frac{2Ab'_0}{B}}{b'_1\exp(\varepsilon\sqrt{A}\xi+\omega)+b'_0+\frac{1}{4}\frac{b'^2_0\Delta}{b'_1B^2}\exp(-\varepsilon\sqrt{A}\xi-\omega)}. \qquad (22)$$

where $\Delta=B^2-4AC$.

Since ω is an arbitrary constant, we set $\omega=0$. Let $\Delta=0$, $A>0$ and $b'_0=b'_1$, then Eq. (22) is reduced to

$$\phi_{11}(\xi)=-\frac{A}{B}[1+\varepsilon\tanh(\frac{\sqrt{A}}{2}\xi)], \qquad (23)$$

If $A>0$, $b'_0=-2B$ and $b'_1=1$, then Eq. (22) becomes

$$\phi_{13}(\xi)=\frac{4A\exp(\varepsilon\sqrt{A}\xi)}{(\exp(\varepsilon\sqrt{A}\xi)-B)^2-4AC}. \qquad (24)$$

Substituting (20) into Eq. (18), we obtain the following general solution of Eq. (17)

$$\phi(\xi)=\frac{a'_0}{b'_1\exp(\varepsilon\sqrt{A}\xi+\omega)-\frac{1}{4}\frac{Ca'^2_0}{Ab'_1}\exp(-\varepsilon\sqrt{A}\xi-\omega)}. \qquad (25)$$

We consider the special case that $\omega=0$. If $A>0$ and we set $b'_1=-\varepsilon Ca'_0$, then Eq. (25), becomes

$$\phi_{14}(\xi)=\frac{4A\varepsilon\exp(\varepsilon\sqrt{A}\xi)}{1-4AC\exp(2\varepsilon\sqrt{A}\xi)}. \qquad (26)$$

Solutions (23), (24) and (26) can be found in [35]. We have obtained all of the periodic and soliton solutions presented in Table 1 in [35] by assigning different values to the free parameters of Eqs. (22) and (25). Hence general solutions given by Eqs. (22) and (25) are very useful for calculating exact solutions of the auxiliary equation.

For more investigation we substitute (21) into Eq. (18). In this case we obtain the general solution to Eq. (17) as

$$\phi(\xi)=\frac{-2\frac{A}{B}b'_0}{b'_1\exp(\varepsilon\sqrt{A}\xi+\omega)+b'_0+\frac{b'^2_0}{4b'_1}\exp(-\varepsilon\sqrt{A}\xi-\omega)}. \qquad (27)$$

As before, let $\omega=0$. Now, if $A>0$ and we set $b'_0=2b'_1$, then Eq. (27) is reduced to

$$\phi_{15}(\xi)=-\frac{A}{B}\mathrm{sech}^2(\frac{1}{2}\varepsilon\sqrt{A}\xi). \qquad (28)$$

If $A > 0$ and we set $b'_0 = -2b'_1$, then Eq. (27) becomes

$$\phi_{16}(\xi) = \frac{A}{B}\mathrm{csch}^2(\frac{1}{2}\varepsilon\sqrt{A}\xi). \tag{29}$$

If $A < 0$ and we set $b'_0 = 2b'_1$, then from Eq. (27) we have

$$\phi_{17}(\xi) = -\frac{A}{B}\sec^2(\frac{1}{2}\varepsilon\sqrt{-A}\xi). \tag{30}$$

If $A < 0$ and we set $b'_0 = -2b'_1$, then Eq. (27) becomes

$$\phi_{18}(\xi) = -\frac{A}{B}\csc^2(\frac{1}{2}\varepsilon\sqrt{-A}\xi). \tag{31}$$

In the next section, we will use Eqs. (1) and (2), as a sub-equation to construct new and more general solution of Eq. (3).

3. More and general exact solutions of the coupled KdV equation

Let use consider the coupled KdV equation (3). By considering the highest order partial derivative with nonlinear term, we suppose that Eq. (3), have the following formal solutions [26]

$$u = A_0 + A_1\phi(\xi) + A_2\phi^2(\xi), \tag{32}$$

$$v = B_0 + B_1\phi(\xi), \tag{33}$$

where A_0, A_1, A_2, B_0 and B_1 are real constants to be determined later, and $\phi(\xi)$ satisfies the general elliptic equation (1), or the auxiliary equation (2), $\xi = ax + bt$, a is a nonzero constant and b is a constant to determined letter. We substitute Eqs. (32) and (33) along with Eq. (1), into Eq. (3), the left-hand sides of Eqs. (3), are converted into two polynomials of $\phi^k(\sqrt{\sum_{p=0}^{4} h_p\phi^p})$, $(k = 0, 1, 2, 3)$. Setting to zero these coefficients, a set of algebraic equations with respect A_0, A_1, A_2, B_0, B_1, a and b is obtained. For more details about this process see [26]. Solving the obtained set of algebraic equations by use of Maple, we get the following solutions:

Case 1. If $h_0 = r^2$, $h_1 = 2rp$, $h_2 = 2qr + p^2$, $h_3 = 2pq$, $h_4 = q^2$, $qr \neq 0$ and $pq \neq 0$, then

$$b=\frac{1}{3}\frac{4a^4qrp^2-a^4p^4-2B_0^2}{ap^2},\ A_0=-\frac{2}{9}\frac{5a^4qrp^2+a^4p^4-B_0^2}{a^2p^2},$$

$$A_1=-2a^2pq,\ A_2=-2a^2q^2,\ B_1=\frac{2B_0q}{p},$$

where a, B_0, p, q and r are arbitrary constants. Substituting the latter equations into Eqs. (32) and (33), we obtain the following solutions of Eq. (3)

$$u(x,t)=-\frac{2}{9}\frac{5a^4qrp^2+a^4p^4-B_0^2}{a^2p^2}-2a^2pq\phi(\xi)-2a^2q^2\phi^2(\xi), \tag{34}$$

and

$$v(x,t)=B_0+\frac{2B_0q}{p}\phi(\xi), \tag{35}$$

where ϕ is a solution obtained in subsection 2.1, $\xi=ax+\frac{1}{3}\frac{4a^4qrp^2-a^4p^4-2B_0^2}{ap^2}t$, $k=\frac{1}{2}\sqrt{p^2-4rq}$, and a_1, a_{-1}, a, B_0, p, q, r, ω are arbitrary constants.

Case 2. If $h_0=h_1=0,\ h_2=A,\ h_3=B,\ h_4=C$, then

$$A_0=-\frac{1}{3}\frac{b+a^3A}{a},\ A_1=-a^2B,\ A_2=-2a^2C,\ B_1=\frac{1}{2}\sqrt{-24aCb+16a^4CA-6a^4B^2},$$

$$B_0=-\frac{aB(3b+a^3A)}{\sqrt{-24aCb+16a^4CA-6a^4B^2}},$$

where a, b, A, $B\neq 0$ and C are arbitrary constants. Substituting these equations into Eqs. (32) and (33), we get the following solutions of Eq. (3)

$$u(x,t)=-\frac{1}{3}\frac{b+a^3A}{a}-a^2B\phi(\xi)-2a^2C\phi^2(\xi), \tag{36}$$

and

$$v(x,t)=-\frac{aB(3b+a^3A)}{\sqrt{-24aCb+16a^4CA-6a^4B^2}}+\frac{1}{2}\sqrt{-24aCb+16a^4CA-6a^4B^2}\phi(\xi), \tag{37}$$

where $\phi(\xi)$ is given by Eq. (22) $\xi=ax+bt$, $\Delta=B^2-4AC$, and b_1', b_0', a, b, A, $B\neq 0$, C, ω are arbitrary constants.

Case 3. If $h_0 = h_1 = h_3 = 0,\ h_2 = A,\ h_4 = C$, then

$$A_0 = -\frac{1}{18}\frac{10a^4AC - B_1^2}{a^2C},\ A_1 = 0,\ A_2 = -2a^2C,\ B_0 = 0,\ b = \frac{1}{6}\frac{4a^4AC - B_1^2}{aC},$$

where a, B_1, $A \neq 0$ and $C \neq 0$ are arbitrary constants. Substituting these relations into Eqs. (32) and (33), we obtain the following solutions of Eq. (3)

$$u(x,t) = -\frac{1}{18}\frac{10a^4AC - B_1^2}{a^2C} - 2a^2C\phi^2(\xi), \tag{38}$$

and

$$v(x,t) = B_1\phi(\xi), \tag{39}$$

where $\phi(\xi)$ is given by Eq. (25), $\xi = ax + \frac{1}{6}\frac{4a^4AC-B_1^2}{aC}t$, and B_1, b_1', a_0', a, $A \neq 0$, $c \neq 0$, ω are arbitrary constants.

To our knowledge, all the solutions reported in this section have not been reported in the literature.

4. Conclusion

In this paper, we applied the Exp-function method for constructing exact solutions of general elliptic and auxiliary equations. We have shown that the obtained general solutions in special cases give many solutions reported in the previous papers. Using these solutions we have constructed exact solutions for coupled KdV equation. We believe that these solutions can be useful for mathematicians and physicians who work in the field of soliton theory.

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