Convergence of the Variational Iteration Method for the Telegraph Equation with Integral Conditions

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Abstract

In this paper, we first transform the telegraph equation into a system of partial differential equations. Then, we apply the variational iteration method to compute an approximate solution for the telegraph equation. Convergence of the proposed method is also discussed. Finally, some numerical examples are given to show the effectiveness of the proposed method.

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1 Introduction

Mathematical modeling of many physical systems lead to ordinary or partial differential equations with non-local conditions in various fields of physics, ecology, biology, etc [4, 6, 8, 9, 10, 31, 34, 38]. Non-local conditions appear when values of the function on the boundary are connected to values inside the domain. In recent years, several numerical techniques have been presented to solve various types of non-local boundary value problems [2, 5, 11, 12, 13, 14, 15].

In the this paper, we consider the telegraph equation

$$\frac{\partial^2 v}{\partial t^2} + \alpha\frac{\partial v}{\partial t} = \beta^2\frac{\partial^2 v}{\partial x^2} + \gamma v + F(x,t), \qquad (x,t) \in \Omega = (0,\ell) \times (0,T], \tag{1}$$

with the initial conditions

$$\begin{aligned} v(x,0) &= r(x), \qquad 0 \le x \le \ell, \\ v_t(x,0) &= s(x), \qquad 0 \le x \le \ell, \end{aligned} \tag{2}$$

the Neumann condition

$$v_x(0,t) = p(t), \qquad 0 < t \le T, \tag{3}$$

and the integral (non-local) condition

$$\int_0^\ell v(x,t)dx = q(t), \qquad 0 < t \le T, \tag{4}$$

where F, r, s, p and q are given functions and $\alpha > 0$, $\gamma < 0$, $\ell > 0$, $T > 0$ and $\beta \in \mathbb{R}$. The existence and uniqueness of the solution of this problem were discussed in [20]. Throughout this paper, we assume that F is sufficiently smooth to produce a smooth classical solution v. Here we mention that the functions r and s satisfy the following compatibility conditions

$$\begin{aligned} r'(0) &= p(0), \qquad \int_0^\ell r(x)dx = q(0), \\ s'(0) &= p'(0), \qquad \int_0^\ell s(x)dx = q'(0). \end{aligned}$$

As we know, the He's variational iteration method (VIM) [21, 22, 23, 24, 25, 26] is a powerful device for solving various kinds of problems. In [1], Ateş and Yildirim applied the VIM to solve fractional initial-value problems. Chun in [7] used this method to solve heat equations with ill-conditioned initial data. Application of the VIM to higher-order nonlinear boundary problems is investigated by Noor and Mohyud-Din in [32]. In [33], Ozer used the VIM to the boundary value problems with jump discontinuities arising in solid mechanics. Solving nonlinear multi-point boundary value problems by combining homotopy perturbation and variational iteration methods is investigated by Geng and Cui [19]. Application of the VIM to integral equations is studied in [43]. It has been shown that the VIM is an efficient algorithm for solving integro-differential equation (see for example [36, 39, 42]). In [3], Biazar et al. applied the VIM to the telegraph equation with some initial conditions. A lot of papers have been written concerning the VIM and we have reviewed only a few of them. For a comprehensive study of the method and its applications the reader is referred to [27, 28, 29].

The convergence of the VIM is systematically discussed by Tatari and Dehghan [40]. Recently, Salkuyeh in [37] showed the convergence of the VIM for linear systems of ordinary differential equations with constant coefficients. Similarly, Torvattanabun and Koonprasert in [41] showed the convergence of the VIM for solving a first-order linear system of PDEs with constant coefficients. In [14], Dehghan and Saadatmandi applied the VIM to solve a special case of problem (1)-(4), without giving any discussion about the convergence of the method. In this paper, we use the VIM to solve problem (1)-(4) and our emphasis is on verifying the convergence of the proposed method.

This paper is organized as follows. In section 2, we give a brief description of the VIM. In section 3, we transform the telegraph equation to another equivalent equation with homogeneous boundary conditions. Section 4, is devoted to the application of the VIM to the telegraph equation and verifying its convergence. In section 5, some numerical examples are given. Some concluding remarks are given in section 6.

2 A brief description of the variational iteration method

Consider the following differential equation

$$\mathcal{L}u(t) + \mathcal{N}u(t) = g(t),$$

where $\mathcal{L}$ is a linear operator, $\mathcal{N}$ a nonlinear operator and $g(t)$ is an inhomogeneous term. In the VIM, a correctional functional as

$$u_{m+1}(t) = u_m(t) + \int_0^t \lambda(\mathcal{L}u_m(s) + \mathcal{N}\widetilde{u}_m(s) - g(s))ds, \quad m = 0, 1, 2, \cdots,$$

is made, where λ is a general Lagrangian multiplier [30] which can be identified optimally via the variational theory. Obviously the successive approximations u_j, $j = 0, 1, \ldots$, can be computed by determining λ. Here, the function $\widetilde{u}_m$ is a restricted variation which means $\delta\widetilde{u}_m = 0$.

3 Reformulation of the problem

For the sake of simplicity, we transform problem (1)-(4) with inhomogeneous conditions (3) and (4) to an equivalent one with homogenous conditions. To do so, we use the transformation [20]

$$u(x,t) = v(x,t) - z(x,t), \qquad (x,t) \in \Omega = (0,\ell) \times (0,T],$$

where

$$z(x,t) = p(t)(x - \frac{\ell}{2}) + \frac{q(t)}{\ell}.$$

In this case, by a simple manipulation the problem is transformed to

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial u}{\partial t} = \beta^2\frac{\partial^2 u}{\partial x^2} + \gamma u + \bar{F}(x,t), \qquad (x,t) \in \Omega = (0,\ell) \times (0,T], \tag{5}$$

with the initial conditions

$$\begin{aligned} u(x,0) &= \bar{r}(x), \qquad 0 \le x \le \ell, \\ u_t(x,0) &= \bar{s}(x), \qquad 0 \le x \le \ell, \end{aligned} \tag{6}$$

the Neumann condition

$$u_x(0,t) = 0, \qquad 0 < t \le T, \tag{7}$$

and the integral condition

$$\int_0^\ell u(x,t)dx = 0, \qquad 0 < t \leq T, \tag{8}$$

where

$$\begin{aligned}\bar{F}(x,t) &= F(x,t) - \frac{\partial^2 z}{\partial t^2} - \alpha\frac{\partial z}{\partial t} + \gamma z,\\ \bar{r}(x) &= r(x) - z(x,0),\end{aligned}$$

and

$$\bar{s}(x) = s(x) - \frac{\partial z}{\partial t}(x,0).$$

As we observe, the Neumann and integral conditions are now homogeneous. Hence, instead of looking for v, we simply look for u. It is easy to see that the compatibility conditions are reduced to

$$\begin{aligned}\bar{r}'(0) &= 0, \qquad \int_0^\ell \bar{r}(x)dx = 0,\\ \bar{s}'(0) &= 0, \qquad \int_0^\ell \bar{s}(x)dx = 0.\end{aligned}$$

After computing u, the solution of problem (1)-(4) will be directly obtained by the relation $v(x,t) = u(x,t) + z(x,t)$.

4 Convergence of the VIM for the telegraph equation

In this section, we focus our attention on the problem (5)-(8). By using the transformation

$$w = \frac{\partial u}{\partial t},$$

the problem can be written as the following system of partial differential equations with constant coefficients

$$\begin{aligned}\frac{\partial u}{\partial t} &= w, && (x,t) \in \Omega = (0,\ell) \times (0,T],\\ \frac{\partial w}{\partial t} &= \gamma u - \alpha w + \beta^2\frac{\partial^2 u}{\partial x^2} + \bar{F}(x,t), && (x,t) \in \Omega = (0,\ell) \times (0,T],\end{aligned}$$

with initial conditions

$$\begin{aligned}u(x,0) &= \bar{r}(x), \qquad 0 \leq x \leq \ell,\\ w(x,0) &= \bar{s}(x), \qquad 0 \leq x \leq \ell,\end{aligned}$$

and boundary conditions

$$u_x(0,t) = 0, \qquad 0 < t \le T,$$

$$\int_0^\ell u(x,t)dx = 0, \qquad 0 < t \le T.$$

For solving the above problem by means of the VIM, it can be written as the following simple form

$$\frac{\partial U(x,t)}{\partial t} = AU(x,t) + B\frac{\partial^2 U(x,t)}{\partial x^2} + \hat{F}(x,t), \qquad (x,t) \in \Omega = (0,\ell) \times (0,T], \tag{9}$$

where

$$U(x,t) = \begin{pmatrix} u(x,t) \\ w(x,t) \end{pmatrix}, \qquad \hat{F}(x,t) = \begin{pmatrix} 0 \\ \bar{F}(x,t) \end{pmatrix},$$

and $A = (a_{ij})$ and B are constant square matrices as following

$$A = \begin{pmatrix} 0 & 1 \\ \gamma & -\alpha \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ \beta^2 & 0 \end{pmatrix}.$$

Now, we apply the VIM to compute an approximate solution for (9). For solving the problem (9) by means of the VIM, the matrix A is decomposed into two matrices D and N such that $A = D + N$, where $D = \text{diag}(0, -\alpha)$ and $N = A - D$. Then, we construct the following correction functional for U

$$\begin{aligned} U_{m+1}(x,t) &= U_m(x,t) + \int_0^t \Lambda(s)[\frac{\partial U_m(x,s)}{\partial s} - B\frac{\partial^2 \tilde{U}_m(x,s)}{\partial x^2} - DU_m(x,s) \\ &\quad -N\tilde{U}_m(x,s) - \hat{F}(x,s)]ds, \end{aligned} \tag{10}$$

where $\Lambda(s) = \text{diag}(\lambda_1(s), \lambda_2(s))$, with $\lambda_1(s)$ and $\lambda_2(s)$ being the Lagrange multipliers and $\tilde{U}_m(x,t)$ being the restricted variation, i.e., $\delta\tilde{U}_m(x,t) = 0$. Note that although $B\frac{\partial^2 \tilde{U}_m(x,s)}{\partial x^2}$ and $N\tilde{U}_m(x,s)$ are not nonlinear terms, we consider it as nonlinear terms. The variation of (10) is then

$$\begin{aligned} \delta U_{m+1}(x,t) &= \delta U_m(x,t) + \delta\int_0^t \Lambda(s)[\frac{\partial U_m(x,s)}{\partial s} - B\frac{\partial^2 \tilde{U}_m(x,s)}{\partial x^2} - DU_m(x,s) \\ &\quad -N\tilde{U}_m(x,s) - \hat{F}(x,s)]ds. \end{aligned}$$

By using integration by parts and constructing the correction functional

$$\begin{aligned} \delta U_{m+1}(x,t) &= \delta U_m(x,t) + \Lambda(s)\delta U_m(x,t)|_{s=t} - \delta\int_0^t [\Lambda'(s)U_m(x,t) \\ &\quad +\Lambda(s)DU_m(x,s) + \Lambda(s)(B\frac{\partial^2 \tilde{U}_m(x,s)}{\partial x^2} + N\tilde{U}_m(x,s) \\ &\quad +\hat{F}(x,s))]ds \\ &= (I + \Lambda(s)|_{s=t})\delta U_m(x,t) - \delta\int_0^t (\Lambda'(s) + \Lambda(s)D)U_m(x,s) \\ &\quad +\Lambda(s)(B\frac{\partial^2 \tilde{U}_m(x,s)}{\partial x^2} + N\tilde{U}_m(x,s) + \hat{F}(x,s))]ds \end{aligned}$$

the stationary conditions would be as follows

$$\begin{aligned} I + \Lambda(s)|_{s=t} &= 0, \\ \Lambda'(s) + \Lambda(s)D &= 0. \end{aligned}$$

Here, the prime stands for differentiation with respect to s. Since $a_{11} = 0$ and $a_{22} = -\alpha$, form the latter equations we obtain $\lambda_1(s) = -1$ and $\lambda_2(s) = -e^{\alpha(s-t)}$. Hence, we have $\Lambda(s) = -e^{-(s-t)D}$. Therefore, from (10) the following iteration formula for computing U_m may be obtained

$$\begin{aligned} U_{m+1}(x,t) = U_m(x,t) - \int_0^t e^{-(s-t)D}[\frac{\partial U_m(x,s)}{\partial s} - B\frac{\partial^2 U_m(x,s)}{\partial x^2} \\ -AU_m(x,s) - \hat{F}(x,s)]ds. \end{aligned} \quad (11)$$

Now, we show that the sequence $U_m(x,t)$ defined by (11) with suitable initial approximation converges to the solution of (9). To do this, we state and prove the following theorem.

Theorem 1. Let $\bar{\Omega} = [0,\ell] \times [0,T]$ and $U(x,t) \in (C^1(\bar{\Omega}))^2$ be the exact solution of (9) and $U_m(x,t) \in (C^1(\bar{\Omega}))^2$ be the solutions of the sequence defined by (11) with $U_0(x,t) = (\bar{r}(x) + t\bar{s}(x), \bar{s}(x))^T$. If $E_m(x,t) = U_m(x,t) - U(x,t)$ and $\|\frac{\partial^2}{\partial x^2}E_m(x,t)\|_2 \le \|E_m(x,t)\|_2$, then the functional sequence defined by (11) converges to $U(x,t)$.

Proof. We first mention that the initial approximation $U_0(x,t)$ satisfies equations (6)-(8). Since $U(x,t)$ is the exact solution of (9), it is obvious that

$$U(x,t) = U(x,t) - \int_0^t e^{-(s-t)D}[\frac{\partial}{\partial s}U(x,s) - B\frac{\partial^2}{\partial x^2}U(x,s) - AU(x,s) - \hat{F}(x,s)]ds, \quad (12)$$

Now from (11) and (12), we get

$$E_{m+1}(x,t) = E_m(x,t) - \int_0^t e^{-(s-t)D}[\frac{\partial}{\partial s}E_m(x,s) - B\frac{\partial^2}{\partial x^2}E_m(x,s) - AE_m(x,s)]ds, \quad (13)$$

By using integration by parts we conclude that

$$\begin{aligned} E_{m+1}(x,t) &= E_m(x,t) - [e^{-(s-t)D}E_m(x,s)|_0^t + \int_0^t e^{-(s-t)D}DE_m(x,s)ds] \\ &+ \int_0^t e^{-(s-t)D}B\frac{\partial^2}{\partial x^2}E_m(x,s)ds + \int_0^t e^{-(s-t)D}AE_m(x,s)ds. \end{aligned}$$

Obviously $E_m(x,0) = 0$, $m = 0, 1, \ldots$. Hence

$$E_{m+1}(x,t) = \int_0^t e^{-(s-t)D}NE_m(x,s)ds + \int_0^t e^{-(s-t)D}B\frac{\partial^2}{\partial x^2}E_m(x,s)ds.$$

Taking 2-norm of both sides of the latter equation gives

$$\begin{aligned}
\|E_{m+1}(x,t)\|_2 &\le \int_0^t \|e^{-(s-t)D}\|_2 \|N\|_2 \|E_m(x,s)\|_2 ds \\
&\quad + \int_0^t \|e^{-(s-t)D}\|_2 \|B\|_2 \|\frac{\partial^2}{\partial x^2} E_m(x,s)\|_2 ds \\
&= \|N\|_2 \int_0^t \|e^{-(s-t)D}\|_2 \|E_m(x,s)\|_2 ds \\
&\quad + \|B\|_2 \int_0^t \|e^{-(s-t)D}\|_2 \|\frac{\partial^2}{\partial x^2} E_m(x,s)\|_2 ds.
\end{aligned}$$

Here we mention that the used matrix norm is the matrix norm induced by the vector 2-norm Now from the assumption $\|\frac{\partial^2}{\partial x^2} E_m(x,t)\|_2 \le \|E_m(x,t)\|_2$, we obtain

$$\begin{aligned}
\|E_{m+1}(x,t)\|_2 &\le \|N\|_2 \int_0^t \|e^{-(s-t)D}\|_2 \|E_m(x,s)\|_2 ds \\
&\quad + \|B\|_2 \int_0^t \|e^{-(s-t)D}\|_2 \|E_m(x,s)\|_2 ds. \qquad (14)
\end{aligned}$$

It is easy to see that $\|B\|_2 = \beta^2$ and $\|N\|_2 = \bar{\gamma} = \max\{1, |\gamma|\}$ and from $s \le t \le T$, we obtain

$$\|e^{-(s-t)D}\|_2 \le e^{\|-(s-t)D\|_2} = e^{\|-(s-t)\|_2 \|D\|_2} \le e^{2t\alpha} \le e^{2T\alpha}$$

Therefore, it follows from (14) that

$$\|E_{m+1}(x,t)\|_2 \le M \int_0^t \|E_m(x,s)\|_2 ds,$$

where $M = (\bar{\gamma} + \beta^2) e^{2T\alpha}$. Now, we proceed as follows

$$\begin{aligned}
\|E_1(x,t)\|_2 &\le M \int_0^t \|E_0(x,s)\|_2 ds \le M \max_{(x,s)\in\bar{\Omega}} \|E_0(x,s)\|_2 \int_0^t ds \\
&= M \max_{(x,s)\in\bar{\Omega}} \|E_0(x,s)\|_2 t, \\
\|E_2(x,t)\|_2 &\le M \int_0^t \|E_1(x,s)\|_2 ds \le M^2 \int_0^t \max_{(x,s)\in\bar{\Omega}} \|E_0(x,s)\|_2 s ds \\
&\le M^2 \max_{(x,s)\in\bar{\Omega}} \|E_0(x,s)\|_2 \frac{t^2}{2!}, \\
&\vdots \\
\|E_m(x,t)\|_2 &\le M \int_0^t \|E_{m-1}(x,s)\|_2 ds \le M^m \int_0^t \max_{(x,s)\in\bar{\Omega}} \|E_0(x,s)\|_2 \frac{s^{m-1}}{(m-1)!} ds \\
&= \max_{(x,s)\in\bar{\Omega}} \|E_0(x,s)\|_2 \frac{(Mt)^m}{m!}.
\end{aligned}$$

Now, we have

$$\max_{(x,s)\in\bar{\Omega}} \|E_0(x,s)\|_2 \frac{(Mt)^m}{m!} \leq \max_{(x,s)\in\bar{\Omega}} \|E_0(x,s)\|_2 \frac{(MT)^m}{m!} \to 0$$

as $m \to 0$, and this completes the proof. □

5 Numerical Examples

In this section, we present some examples to show the efficiency of the proposed method for solving problem (1)-(4). All of the computations have done by the MAPLE software.

Example 1. For the first example we consider

$$\frac{\partial^2}{\partial t^2} v(x,t) + 8\frac{\partial}{\partial t} v(x,t) = \frac{\partial^2}{\partial x^2} v(x,t) - 4v(x,t) + F(x,t),$$

where $(x,t) \in \Omega = (0, 2\pi) \times (0, 0.5]$, and

$$\begin{aligned} F(x,t) &= -2e^{-t}\sin x, \quad 0 \leq x \leq 2\pi, \quad 0 < t \leq 0.5, \\ r(x) &= \sin x, \quad s(x) = -\sin x, \quad 0 \leq x \leq 2\pi, \\ p(t) &= e^{-t}, \quad q(t) = 0, \quad 0 < t \leq 0.5. \end{aligned}$$

The exact solution of this problem is $v(x,t) = e^{-t}\sin x$ [18, 16, 35]. For this problem, we obtain

$$\begin{aligned} z(x,t) &= e^{-t}(x - \pi), \\ \bar{F}(x,t) &= -2e^{-t}\sin x + 3e^{-t}(x - \pi), \\ \bar{r}(x) &= \sin x - x + \pi, \\ \bar{s}(x) &= -\sin x + x - \pi. \end{aligned}$$

Now, we can apply the proposed method in the previous section to obtain the approximate solution of the problem. Based on Theorem 1, we use

$$U_0(x,t) = \begin{pmatrix} u_0(x,t) \\ w_0(x,t) \end{pmatrix} = \begin{pmatrix} (1-t)(\sin(x) - x + \pi) \\ -\sin(x) + x - \pi \end{pmatrix},$$

as the initial guess. To show the convergence behavior of the sequence (11), the values of $\|v - v_m\|_1$ for different values of m are given in Table 1, where $v_m = u_m + z$. As the numerical results in this table show the proposed method is very effective. For more investigation, the absolute error

$$E(x,t) = |v(x,t) - v_{12}(x,t)|, \qquad (x,t) \in \Omega = (0, 2\pi) \times (0, 0.5].$$

is plotted in Figure 1. As we observe, there is very good agreement between the approximate solution obtained by the VIM and the exact solution.

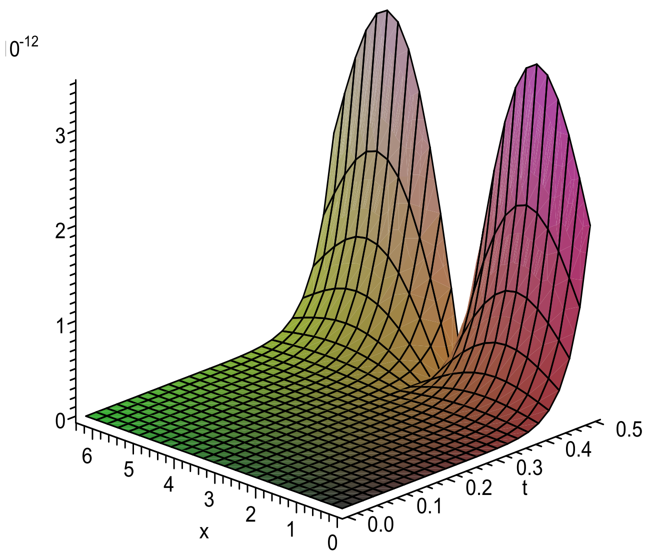

Figure 1: Depiction of the absolute error for Example 1 with $m = 12$.

Table 1: Numerical results for the Example 1.

m	2	4	6	8	10	12
$\|v - v_m\|_1$	8.293e-3	1.592e-4	1.998e-6	1.761e-8	1.149e-10	5.955e-13

Example 2. Consider

$$\frac{\partial^2}{\partial t^2}v(x,t) + 12\frac{\partial}{\partial t}v(x,t) = \frac{\partial^2}{\partial x^2}v(x,t) - 4v(x,t) + F(x,t),$$

where $(x,t) \in \Omega = (0,\pi) \times (0,.5]$, with

$$\begin{aligned} F(x,t) &= 4\sin x(\cos t - 3\sin t), \quad 0 \le x \le \pi, \quad 0 < t \le 0.5, \\ r(x) &= \sin x, \quad s(x) = 0, \quad 0 \le x \le \pi, \\ p(t) &= \cos t, \quad q(t) = 2\cos t, \quad 0 < t \le 0.5. \end{aligned}$$

The exact solution of this problem is $v(x,t) = \cos t \sin x$ [17, 18]. For this problem, we obtain

$$\begin{aligned} z(x,t) &= \frac{2\pi x - \pi^2 + 4}{2\pi}\cos(t), \\ \bar{F}(x,t) &= 4\sin x(\cos t - 3\sin t) - \frac{3(2\pi x - \pi^2 + 4)}{2\pi}\cos t + \frac{6(2\pi x - \pi^2 + 4)}{\pi}\sin t, \\ \bar{r}(x) &= \sin x - \frac{2\pi x - \pi^2 + 4}{2\pi}, \\ \bar{s}(x) &= 0. \end{aligned}$$

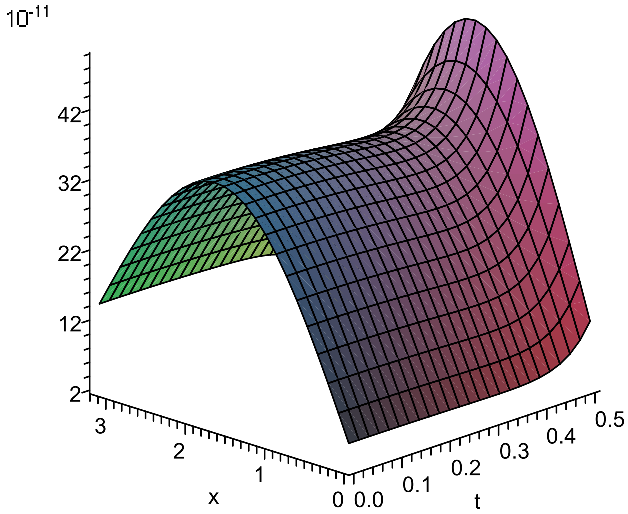

Figure 2: Depiction of the absolute error for Example 2 with $m = 10$.

Here, we use

$$U_0(x,t) = \begin{pmatrix} u_0(x,t) \\ w_0(x,t) \end{pmatrix} = \begin{pmatrix} \sin(x) - \frac{2\pi x - \pi^2 + 4}{2\pi} \\ 0 \end{pmatrix},$$

as the initial guess. Similar to the previous example, the values of $\|v - v_m\|_1$ for different values of m are given in Table 2, and the absolute error $E(x,t) = |v(x,t) - v_{10}(x,t)|$ is plotted in Figure 2. As we see, the proposed method is very effective.

Table 2: Numerical results for the Example 2.

m	2	4	6	8	10
$\|v - v_m\|_1$	1.376e-3	2.327e-5	2.766e-7	3.621e-9	4.204e-10

Example 3. Consider the problem

$$\frac{\partial^2}{\partial t^2} v(x,t) + 2\frac{\partial}{\partial t} v(x,t) = \frac{\partial^2}{\partial x^2} v(x,t) - v(x,t) + F(x,t),$$

where $(x,t) \in \Omega = (0,4) \times (0,1]$, with

$$\begin{aligned} F(x,t) &= 2(t^2 - x - x^2)e^{-t}, \quad 0 \le x \le 4, \quad 0 < t \le 1, \\ r(x) &= 0, \quad s(x) = 0, \quad 0 \le x \le 4, \\ p(t) &= -t^2 e^{-t}, \quad q(t) = -\frac{88}{3} t^2 e^{-t}, \quad 0 < t \le 1. \end{aligned}$$

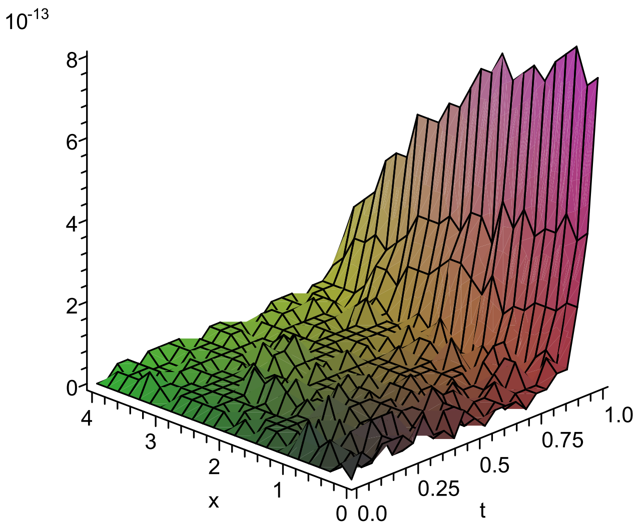

Figure 3: Depiction of the absolute error for Example 1 with $m = 14$.

The exact solution of this problem is $v(x,t) = -(x^2 + x)t^2 e^{-t}$. Here, we have

$$\begin{aligned} z(x,t) &= -\frac{1}{3}t^2(3x+16)e^{-t}, \\ \bar{F}(x,t) &= \frac{2}{3}(-3x^2+3t^2+16)e^{-t}, \\ \bar{r}(x) &= 0, \quad \bar{s}(x) = 0. \end{aligned}$$

For this example we use the following initial guess

$$U_0(x,t) = \begin{pmatrix} u_0(x,t) \\ w_0(x,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Similar to the previous examples, the values of $\|v - v_m\|_1$ for different values of m are given in Table 3, and the absolute error $E(x,t) = |v(x,t) - v_{14}(x,t)|$ is plotted in Figure 3. As the figure shows the VIM method gives very good results.

Table 3: Numerical results for the Example 3.

m	4	6	8	10	12	14
$\|v - v_m\|_1$	3.737e-3	6.211e-5	6.784e-7	5.216e-9	2.945e-11	2.625e-13

6 Conclusions

In this paper, we showed that under some conditions the variational iteration method is convergent for the telegraph equation with non-local boundary conditions. Numerical results presented in this paper show that the proposed method is very effective. The proposed method gives not only the function v but also its derivative, simultaneously. As Biazar et al. mentioned in [3] and we have observed in this paper, the solution procedure is very simple by means of variational theory, and only a few steps lead to high accurate solution valid for the whole solution domain. The method does not need small parameter or linearization. Moreover, by the VIM one can obtain a sufficiently smooth solution for sufficiently smooth data.

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