A note on the paper "Convergence of the TAGE iterative method for the system arisen from the cubic spline approximation for the solution of two-point BVPs with forcing function in integral form", by Mohanty, Jain and Dhall

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Abstract

In this note, we point out an error in the recently published article (Mohanty et al., Appl. Math. Model. 35 (2011) 3036-3047) and then correct it.

Key words: Cubic spline, boundary value problem, integral form, TAGE iterative method, convergence.
2000 MSC: 65L10, 65F10.

1. Introduction

Recently, Mohanty et al. in [5] proposed a method based on the cubic spline approximation for the solution of two-point nonlinear boundary value problems, whose forcing functions are in integral form. Their method gives a linear system of equations of the form

$$(A_1 + A_2)u = RH, \tag{1}$$

where

$$A_1 = \begin{pmatrix} b_1 & & & & & 0 \\ & b_2 & c_2 & & & \\ & a_3 & b_3 & & & \\ & & & \ddots & & \\ & & & & b_{N-1} & c_{N-1} \\ 0 & & & & a_N & b_N \end{pmatrix}, \quad A_2 = \begin{pmatrix} b_1 & c_1 & & & & 0 \\ a_2 & b_2 & & & & \\ & & \ddots & & & \\ & & & b_{N-2} & c_{N-2} & \\ & & & a_{N-1} & b_{N-1} & \\ 0 & & & & & b_N \end{pmatrix},$$

and

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}, \qquad RH = \begin{pmatrix} RH_1 \\ RH_2 \\ \vdots \\ RH_N \end{pmatrix}.$$

Then, they applied the two parameter alternating group explicit (TAGE) iterative method to solve system (1). This method may be written as

$$\begin{cases} (A_1 + \rho_1 I)v = RH - (A_2 - \rho_1 I)u^{(s)}, \\ (A_2 + \rho_2 I)u^{(s+1)} = RH - (A_1 - \rho_2 I)v, \end{cases} \quad s = 0, 1, 2, \ldots, \tag{2}$$

where $u^{(0)}$ is an initial guess to the solution, v is an auxiliary vector and $\rho_1, \rho_2 > 0$. If $\{v^{(s)}\}$ is the two-step iteration sequence defined by the TAGE iterative method, then

$$u^{(s+1)} = Gu^{(s)} + g, \quad s = 0, 1, \ldots, \tag{3}$$

where the TAGE iteration matrix G is given by

$$G = (A_2 + \rho_2 I)^{-1}(A_1 - \rho_2 I)(A_1 + \rho_1 I)^{-1}(A_2 - \rho_1 I), \tag{4}$$

and

$$g = (A_2 + \rho_2 I)^{-1}[I - (A_1 - \rho_2 I)(A_1 + \rho_1 I)^{-1}]RH.$$

It is well-known that the TAGE iterative method is convergent if and only if $S(G) < 1$, where $S(G)$ denotes the spectral radius of G. Let G^* be a similar matrix to G and is given by

$$\begin{aligned} G^* &= (A_2 + \rho_2 I)G(A_2 + \rho_2 I)^{-1} \\ &= (A_1 - \rho_2 I)(A_1 + \rho_1 I)^{-1}(A_2 - \rho_1 I)(A_2 + \rho_2 I)^{-1}. \end{aligned}$$

Then, we have

$$\|G^*\|_2 \leq \|(A_1 - \rho_2 I)(A_1 + \rho_1 I)^{-1}\|_2 \|(A_2 - \rho_1 I)(A_2 + \rho_2 I)^{-1}\|_2.$$

In [5], the authors concluded that for every $\rho_1, \rho_2 > 0$,

$$\|(A_1 - \rho_2 I)(A_1 + \rho_1 I)^{-1}\|_2 = \max_{\lambda_k \in \sigma(A_1)} |\frac{\lambda_k - \rho_2}{\lambda_k + \rho_1}| < 1, \tag{5}$$

$$\|(A_2 - \rho_1 I)(A_2 + \rho_2 I)^{-1}\|_2 = \max_{\mu_k \in \sigma(A_2)} |\frac{\mu_k - \rho_1}{\mu_k + \rho_2}| < 1, \tag{6}$$

where for a matrix X, $\sigma(X)$ denotes the spectrum of X. Then, they deduced that

$$S(G) = S(G^*) \leq \|G^*\|_2 < 1. \tag{7}$$

In the sequel, we present a counterexample for this conclusion. Consider the matrix $A = A_1 + A_2$ of the form

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -2 & 8 & -4 & 0 & 0 \\ 0 & -1 & 8 & -2 & 0 \\ 0 & 0 & -1 & 10 & -1 \\ 0 & 0 & 0 & -16 & 10 \end{pmatrix}.$$

According to the definition of the matrices A_1 and A_2, we have

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & -16 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

The spectrum of A_1 and A_2 are $\sigma(A_1) = \{1, 2, 6, 9\}$ and $\sigma(A_2) = \{0.4384, 4.5616, 3, 6, 5\}$. As we see the eigenvalues of A_1 and A_2 are positive. For $\rho_1 = 1$ and $\rho_2 = 8$, we have

$$\|(A_1 - \rho_2 I)(A_1 + \rho_1 I)^{-1}\|_2 = 7.6040 > \max_{\lambda_k \in \sigma(A_1)} |\frac{\lambda_k - \rho_2}{\lambda_k + \rho_1}| = 3.5 > 1,$$

and

$$\|(A_2 - \rho_1 I)(A_2 + \rho_2 I)^{-1}\|_2 = 0.3635 > \max_{\mu_k \in \sigma(A_2)} |\frac{\mu_k - \rho_1}{\mu_k + \rho_2}| = 0.3571.$$

Moreover, we have

$$\|(A_1 - \rho_2 I)(A_1 + \rho_1 I)^{-1}\|_2 \|(A_2 - \rho_1 I)(A_2 + \rho_2 I)^{-1}\|_2 = 2.7638 > 1,$$

$$\max_{\lambda_k} |\frac{\mu_k - \rho_2}{\mu_k + \rho_1}| \max_{\mu_k} |\frac{\mu_k - \rho_2}{\mu_k + \rho_1}| = 1.25 > 1.$$

We observe that, Eq. (7) is not correct for all positive parameters ρ_1 and ρ_2 and the convergence of the TAGE method to solve system (1) does not follow. In the next section we prove the convergence of the method under some additional assumptions.

2. Convergence of the TAGE iterative method for system (1)

We first state and prove the following lemma.

Lemma 1. *Let h_k be sufficiently small. Then, the eigenvalues of A_1 and A_2 are all real.*

Proof. Let h_k be sufficiently small. According to the definition of a_k's and c_k's (see [5]), we have

$$\begin{aligned} a_k &= -\sigma_k - \frac{h_k}{12}[R_k D_k^{**} + \frac{\sigma_k^2 Q_k D_k}{S_k}] - \frac{h_k^2}{72(1+\sigma_k)}\left[\sigma_k\beta_k D_k - \sigma^2\gamma_k D_k^* - (2+\sigma_k)\eta_k D_k^{**}\right] \\ &\quad + \frac{h_k^2}{72}[6R_k - h_k\eta_k]E_k^{**} = -\sigma_k - \mathcal{O}(h_k) < 0, \\ c_k &= -1 + \frac{h_k}{12\sigma_k}[P_k D_k^* + \frac{\sigma_k Q_k D_k}{S_k}] + \frac{h_k^2}{72 S_k}[\beta_k D_k + \sigma_k(1+2\sigma_k)\gamma_k D_k^* + \eta_k D_k^{**}] \\ &\quad + \frac{h_k^2}{72}[6P_k + \sigma_k h_k \gamma_k]E_k^* = -1 + \mathcal{O}(h_k) < 0. \end{aligned}$$

Here, we mention that $(\sqrt{5}-1)/2 < \sigma_k < (\sqrt{5}+1)/2$. As a result, we have $a_{k+1}c_k > 0$, $k = 2, 4, \ldots, N-1$. Now, let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the eigenvalues of A. Evidently, $\lambda_1 = b_1$ and $\lambda_2, \lambda_3, \ldots, \lambda_N$ are the roots of the quadratic equations

$$\lambda^2 - (b_k + b_{k+1})\lambda + (b_k b_{k+1} - a_{k+1}c_k) = 0, \quad k = 2, 4, \ldots, N-1.$$

It is easy to verify the the discriminant of these quadratic equations are

$$\Delta_k = (b_k - b_{k+1})^2 + 4a_{k+1}c_k > 0, \quad k = 2, 4, \ldots, N-1.$$

This shows that the eigenvalues of A_1 are all real.

In the same way one can see that the eigenvalues of A_2 are all real as well. □

The next theorem gives sufficient conditions for the convergence of the TAGE iterative method to solve system (1)

Theorem 1. *Let λ_i's and μ_i's be the eigenvalues of A_1 and A_2, respectively. If*

$$\rho_1 > \max\{0, -\lambda_1, \ldots, -\lambda_N\}, \tag{8}$$
$$\rho_2 > \max\{0, -\mu_1, \ldots, -\mu_N\}, \tag{9}$$
$$\rho_1 - 2\min_k \mu_k < \rho_2 < \rho_1 + 2\min_k \lambda_k, \tag{10}$$

then the TAGE iterative method is convergent for the system (1).

Proof. Let

$$D = \mathrm{diag}(1, \frac{c_1}{a_2}, \frac{c_1 c_2}{a_2 a_3}, \ldots, \frac{c_1 c_2 \ldots c_{N-1}}{a_2 a_3 \ldots a_N}) \equiv \mathrm{diag}(d_1, d_2, \ldots, d_N).$$

As we have seen in the previous lemma, the off-diagonal entries of A are negative. This shows that $c_k a_{k+1} > 0$, $k = 1, \ldots N-1$. Therefore the diagonal entries of D are positive. We have

$$S(G) = S(G^*) = S(D^{1/2}G^*D^{-1/2}),$$

where $D^{1/2} = \mathrm{diag}(d_1^{1/2}, d_2^{1/2}, \ldots, d_N^{1/2})$. On the other hand, we have

$$D^{1/2}G^*D^{-1/2} = (\bar{A}_1 - \rho_2 I)(\bar{A}_1 + \rho_1 I)^{-1}(\bar{A}_2 - \rho_1 I)(\bar{A}_2 + \rho_2 I)^{-1},$$

where $\bar{A}_1 = D^{1/2}A_1D^{-1/2}$ and $\bar{A}_2 = D^{1/2}A_2D^{-1/2}$. Thus,

$$\begin{aligned} S(G) &= S(G^*) = S(D^{1/2}G^*D^{-1/2}) \le \|D^{1/2}G^*D^{-1/2}\|_2 \\ &\le \|(\bar{A}_1 - \rho_2 I)(\bar{A}_1 + \rho_1 I)^{-1}\|_2 \|(\bar{A}_2 - \rho_1 I)(\bar{A}_2 + \rho_2 I)^{-1}\|_2. \end{aligned}$$

It is easy to see that the matrices $\bar{A}_1$ and $\bar{A}_2$ are symmetric (see [3, page 90]) and similar to A_1 and A_2, respectively. As a result, both of the matrices $(\bar{A}_1 - \rho_2 I)(\bar{A}_1 + \rho_1 I)^{-1}$ and $(\bar{A}_2 - \rho_1 I)(\bar{A}_2 + \rho_2 I)^{-1}$ are symmetric as well. Hence, we can conclude that

$$\begin{aligned} \|(\bar{A}_1 - \rho_2 I)(\bar{A}_1 + \rho_1 I)^{-1}\|_2 &= \max_{\lambda_k \in \sigma(\bar{A}_1)} |\frac{\lambda_k - \rho_2}{\lambda_k + \rho_1}| = \max_{\lambda_k \in \sigma(A_1)} |\frac{\lambda_k - \rho_2}{\lambda_k + \rho_1}|, \\ \|(\bar{A}_2 - \rho_1 I)(\bar{A}_2 + \rho_2 I)^{-1}\|_2 &= \max_{\mu_k \in \sigma(\bar{A}_2)} |\frac{\mu_k - \rho_1}{\mu_k + \rho_2}| = \max_{\mu_k \in \sigma(A_2)} |\frac{\mu_k - \rho_1}{\mu_k + \rho_2}|. \end{aligned}$$

Therefore, we have

$$S(G) \le \delta(\rho_1, \rho_2) = \max_{\lambda_k \in \sigma(A_1)} |\frac{\lambda_k - \rho_2}{\lambda_k + \rho_1}| \max_{\mu_k \in \sigma(A_2)} |\frac{\mu_k - \rho_1}{\mu_k + \rho_2}|.$$

From Eqs. (8) and (9), we see that $\rho_1, \rho_2 > 0$ and $\lambda_k + \rho_1 > 0$ for $k = 1, \ldots, N$. Hence,

$$\frac{\lambda_k - \rho_2}{\lambda_k + \rho_1} < 1, \quad k = 1, \ldots, N. \tag{11}$$

On the other hand, from Eq. (10) we have

$$\rho_2 < \rho_1 + 2\min_k \lambda_k \le \rho_1 + 2\lambda_k, \quad k = 1, \ldots, N.$$

This shows that

$$-1 < \frac{\lambda_k - \rho_2}{\lambda_k + \rho_1}, \quad k = 1, \ldots, N. \tag{12}$$

Now, from Eqs. (11) and (12) we conclude that

$$|\frac{\lambda_k - \rho_2}{\lambda_k + \rho_1}| < 1, \quad k = 1, \ldots, N.$$

Thus,

$$\max_{\lambda_k \in \sigma(A_1)} |\frac{\lambda_k - \rho_2}{\lambda_k + \rho_1}| < 1.$$

In the same way, we have

$$\max_{\mu_k \in \sigma(A_2)} |\frac{\mu_k - \rho_1}{\mu_k + \rho_2}| < 1.$$

Hence, we have $\delta(\rho_1, \rho_2) < 1$ which completes the proof. $\square$

Remark 1. *Let $\lambda_k, \mu_k > 0$, $k = 1, \ldots, n$. For every $\rho_1, \rho_2 > 0$, the TAGE iterative method is convergent for the system (1), if*

$$\rho_1 - 2\min_k \mu_k < \rho_2 < \rho_1 + 2\min_k \lambda_k.$$

Moreover, the method is convergent for all $\rho_1 = \rho_2 = \rho > 0$.

Now, we are going to find the optimal values of parameters ρ_1 and ρ_2, i.e., the values which minimize $\delta(\rho_1, \rho_2)$, in the case that the eigenvalues of A_1 and A_2 are positive . To do so, we write down the TAGE iterative method as

$$\begin{cases} (I + \tau_1 A_1)v = \tau_1 RH + (I - \tau_1 A_2)u^{(s)}, \\ (I + \tau_2 A_2)u^{(s+1)} = \tau_2 RH - (I - \tau_2 A_1)v, \end{cases} \qquad s = 0, 1, \ldots, \tag{13}$$

where $\tau_1 = 1/\rho_1$ and $\tau_2 = 1/\rho_2$. This iterative method is known as the alternating direction iteration (ADI) method [1]. Let

$$\alpha = \min_{k=1,\ldots,N} \{\lambda_k, \mu_k\}, \quad \beta = \max_{k=1,\ldots,N} \{\lambda_k, \mu_k\}.$$

According to a result presented in [1, page 297] for the ADI method it is straightforward to see the optimal parameters for the TAGE iterative method to solve the system (1) are $\rho_1 = \rho_2 = \sqrt{\alpha\beta}$.

3. Acknowledgement

The author is grateful to the anonymous referees for their helpful comments and suggestions which substantially improved the quality of this paper.

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