Convergence of the variational iteration method for solving linear systems of ODEs with constant coefficients

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ABSTRACT

In this paper, He’s variational iteration method is applied for solving linear systems of ordinary differential equations with constant coefficients. A theorem for the convergence of the method is presented. Some illustrative examples are given to show the efficiency of the method.

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1. Introduction

Consider the system of ordinary differential equations (ODEs) with constant coefficients

\[
\begin{align*}
Y'(t) &= AY(t) + F(t), \quad 0 \leq t \leq \ell, \\
Y(0) &= Y_0, 
\end{align*}
\]

(1)

where \( Y(t) = (y_1, y_2, \ldots, y_n)^T \) in which \( y_i, i = 1, \ldots, n \) are unknown real functions of variable \( t \), \( A \in \mathbb{R}^{n \times n} \), \( F(t) = (f_1, f_2, \ldots, f_n)^T \) in which \( f_i, i = 1, \ldots, n \) are known real functions of \( t \) and \( Y_0 \) is a given vector in \( \mathbb{R}^n \). As we know well, the solution of Eq. (1) is of the form (see for example [1])

\[
Y(t) = e^{At}Y_0 + \int_0^t e^{-As}F(s)ds.
\]

(2)

Hence, for computing the solution \( Y(t) \) to Eq. (1), \( e^{At} \) should be computed for some vector function \( Z(t) \). It is well-known that the computation of \( e^{At} \) leads to the eigenvalue problem for the matrix \( A \). Indeed, in general the computation of a matrix exponential leads to the computation of the Jordan canonical form of the matrix which has its own difficulties [2,3]. For a fixed \( t \), there are several methods for computing an approximation of \( e^{At} \): Padé approximation [4]; methods based on the Krylov subspace [5]; restrictive Taylor method [6]. Also, there are several methods for computing the analytical solution for the linear and nonlinear equations. Among them are the homotopy perturbation technique [7–9], Adomian decomposition method [10,11] and the variational iteration method presented by He [12–17]. The variational iteration method is a powerful mathematical tool for finding solutions of linear and nonlinear problems and it can be implemented easily in practice. It has been successfully applied for solving various PDEs and ODEs [18,12–17]. Recently, Darvishi et al. in [18] have applied the variational iteration method for solving systems of linear and nonlinear stiff ODEs. Particularly, they have considered systems

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of stiff ODEs with two equations. In this paper, we apply He's variational iteration method for computing an approximate-
theoretical solution to (1) by generalizing the method presented in [18]. The convergence of the method is also studied.

This paper is organized as follows. In Section 2, we give a brief description of He's variational iteration method. Section 3
is devoted to the proposed method and its convergence. In Section 4, some illustrative examples are given. Some concluding
remarks are given in Section 5.

2. A brief description of the variational iteration method

Consider the following differential equation
\[ \mathcal{L}u(t) + \mathcal{N}u(t) = g(t), \tag{3} \]
where \( \mathcal{L} \) is a linear operator, \( \mathcal{N} \) a nonlinear operator and \( g(t) \) is an inhomogeneous term. In the variational iteration method
a correctional functional such as
\[ u_{m+1}(t) = u_m(t) + \int_0^t \lambda(\mathcal{L}u_m(s) + \mathcal{N}\tilde{u}_m(s) - g(s))\,ds, \quad m = 0, 1, 2, \ldots, \]
is constructed, where \( \lambda \) is a general Lagrangian multiplier [12–17], which can be identified optimally via the variational
theory. Obviously the successive approximations \( u_j, j = 0, 1, \ldots \) can be computed by determining \( \lambda \). Here the function \( \tilde{u}_m \)
is a restricted variation which means \( \delta\tilde{u}_m = 0 \).

3. He's variational iteration method for solving problem (1)

For solving problem (1) by means of the variational iteration method, the matrix \( A = (a_{ij}) \) is decomposed into two
matrices \( D \) and \( B \) such that \( A = D + B \), where \( D = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn}) \) and \( B = A - D \). Then we construct the following
correctional functional for \( Y \)
\[ Y_{m+1}(t) = Y_m(t) + \int_0^t A(Y_m(s) - DY_m(s) - \tilde{B}Y_m(s) - F(s))\,ds, \tag{4} \]
where \( A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \), in which \( \lambda_i, i = 1, 2, \ldots, m \) are the Lagrange multipliers and \( \tilde{Y}_m \) denotes the restrictive
variation, i.e., \( \delta\tilde{Y}_m = (\delta\tilde{Y}_1, \ldots, \delta\tilde{Y}_m)^T = 0 \). Note that although \( BY \) is not a nonlinear term, we consider it as a nonlinear term. By using integration by parts and constructing the correctional functional
\[ \delta Y_{m+1}(t) = \delta Y_m(t) + \delta \int_0^t A(Y'_m(s) - DY_m(s) - \tilde{B}Y_m(s) - F(s))\,ds \]
\[ = \delta Y_m(t) + A(s)\delta Y_m(s)|_{s=t} - \delta \int_0^t (A' + AD)Y_m(s) + A(\tilde{B}Y + F(s))\,ds, \]
the stationary conditions would be as follows
\[ I + A(s)|_{s=t} = 0, \]
\[ A'(s) + A(s)D = 0. \]
Here, the prime stands for differentiation with respect to the \( s \). The latter equations can be written as
\[ \begin{align*}
1 + \lambda_i(s)|_{s=t} &= 0, \\
\lambda_i'(s) + a_{ii}\lambda_i(s) &= 0, \quad i = 1, 2, \ldots, n.
\end{align*} \]
For a fixed \( i \) we consider two cases. If \( a_{ii} = 0 \) then it follows that \( \lambda_i(s) = -1 \) and if \( a_{ii} \neq 0 \), then \( \lambda_i = -e^{-a_{ii}(t-s)} \). Hence we have \( A = -e^{-D(t-s)} \). Therefore, from (4) the following iteration formula for computing \( Y_m(t) \) may be obtained
\[ Y_{m+1}(t) = Y_m(t) - \int_0^t e^{-(t-s)D}(Y'_m(s) - AY_m(s) - F(s))\,ds, \quad m = 0, 1, \ldots. \tag{5} \]

Now, we show that the sequence \( \{Y_m(t)\}_{m=1}^\infty \) defined by (5) with \( Y_0(t) = Y_0 \) converges to the solution of (1). To do this
we state and prove the following theorem.

Theorem 1. Let \( Y(t), Y_i(t) \in (C^1[0, \ell])^n, i = 0, 1, \ldots \). The sequence defined by (5) with \( Y_0(t) = Y_0 \) converges to \( Y(t) \), the
exact solution of (1).
**Proof.** Obviously from (1) we have

\[
Y(t) = Y(t) - \int_0^t e^{-(s-t)D}(Y'(s) - AY(s) - F(s))ds.
\] (6)

Now from (5) and (6) we get

\[
E_{m+1}(t) = E_m(t) - \int_0^t e^{-(s-t)D}(E'_m(s) - AE_m(s))ds,
\] (7)

where \(E_j(t) = Y_j(t) - Y(t), j = 1, 2, \ldots\). By using the fact that \(E_m(0) = 0, m = 0, 1, \ldots\), and integration by parts we conclude

\[
E_{m+1}(t) = E_m(t) - \int_0^t e^{(A-D)+t\partial} \frac{d}{ds}(e^{-sA}E_m(s))ds,
\]

\[
= E_m(t) - \int_0^t e^{(A+\partial)t} \frac{d}{ds}(e^{-sA}E_m(s))ds,
\]

\[
= B \int_0^t e^{-(s-t)D} E_m(s)ds.
\] (8)

Therefore

\[
\|E_{m+1}(t)\|_2 \leq \|B\|_2 \int_0^t \|e^{-(s-t)D}\|_2 \|E_m(s)\|_2 ds.
\] (9)

Since \(s \leq t \leq \ell\), we deduce that

\[
\|e^{-(s-t)D}\|_2 \leq e^{-(s-t)\|D\|_2} = e^{\|s-t\|\|D\|_2} \leq e^{2t \max|\alpha_i|} \leq e^{2\ell \max|\alpha_i|}.
\]

Let

\[
M = \|B\|_2 e^{2\ell \max|\alpha_i|}.
\]

Hence, from (9) we obtain

\[
\|E_{m+1}(t)\|_2 \leq M \int_0^t \|E_m(s)\|_2 ds.
\]

Now we proceed as follows

\[
\|E_1(t)\|_2 \leq M \int_0^t \|E_0(s)\|_2 ds \leq M \max_{s \in [0,\ell]} \|E_0(s)\|_2 \int_0^t ds \leq M \max_{s \in [0,\ell]} \|E_0(s)\|_2 t.
\]

\[
\|E_2(t)\|_2 \leq M \int_0^t \|E_1(s)\|_2 ds = M \int_0^t \max_{s \in [0,\ell]} \|E_0(s)\|_2 ds = M^2 \max_{s \in [0,\ell]} \|E_0(s)\|_2 \frac{\ell^2}{2!},
\]

\[
\|E_3(t)\|_2 \leq M \int_0^t \|E_2(s)\|_2 ds = M \int_0^t \max_{s \in [0,\ell]} \|E_0(s)\|_2 \frac{s^2}{2!} ds = M^3 \max_{s \in [0,\ell]} \|E_0(s)\|_2 \frac{\ell^3}{3!},
\]

\[\vdots\]

\[
\|E_m(t)\|_2 \leq M \int_0^t \|E_{m-1}(s)\|_2 ds = M^m \int_0^t \max_{s \in [0,\ell]} \|E_0(s)\|_2 \frac{s^{m-1}}{(m-1)!} ds
\]

\[
= M^m \max_{s \in [0,\ell]} \|E_0(s)\|_2 \frac{\ell^m}{m!} = \frac{(M\ell)^m}{m!}.
\]

We have

\[
\max_{s \in [0,\ell]} \|E_0(s)\|_2 \frac{(M\ell)^m}{m!} \leq \max_{s \in [0,\ell]} \|E_0(s)\|_2 \frac{(M\ell)^m}{m!} \to 0,
\]

as \(m \to \infty\), and this completes the proof. \(\square\)

In practice, for improving the convergence rate of the method, as in the classical Gauss–Seidel method [3] for solving linear system of equations, as soon as a component of \(Y_{m+1}\) is computed then it is used in computing the next component of \(Y_{m+1}\).


4. Illustrative examples

In this section, three illustrative examples are given to show the efficiency of the method proposed in Section 3. All of the computations have been done using the Maple software. For all of the examples in this section we used 8 iterations of the variational iteration method.

Example 1. In this example we consider the problem (1) with

\[ A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix}, \]

\[ F(t) = 0, \quad Y_0 = (7, 2, -\frac{9}{2})^T. \]

The exact solution of this problem is

\[ Y(t) = \begin{pmatrix} 4e^t + 3(1 + t)e^{2t} \\ e^t + (1 + t)e^{2t} \\ -3e^t - \left(\frac{3}{2} + 2t\right)e^{2t} \end{pmatrix}. \]

By using the proposed method in the previous section the computed approximate solution for the second entry of \( Y(t) \) is as follows

\[
\tilde{y}_2 = \frac{56}{3}t^3e^{4t} - \frac{88471543}{11664}e^{4t} - \frac{315752834}{11390625}e^{-2t} + \frac{42319}{1296}e^{4t} - \frac{7662572}{151875}e^{-2t} - \frac{42}{25}t^2e^{-2t} - \frac{8}{45}t^4e^{-2t} + \frac{212}{90}t^4e^{-2t} - \frac{19505}{54}t^2e^{4t} - \frac{64376}{2025}t^2e^{-2t} - \frac{146734}{3375}t^3e^{-2t} - \frac{212}{27}t^4e^{-2t} + \frac{2618281}{1265625}t^3e^{3t} + \frac{412856648}{84375}t^3e^{3t},
\]

\[ + \frac{293776}{1125}t^3e^{3t} + \frac{8197192}{5625}t^4e^{3t} + \frac{2152}{75}t^4e^{3t} + \frac{48}{25}t^5e^{3t}. \]

The exact and approximate solutions are depicted in Fig. 1. As we see, there is very good agreement between the approximate solution obtained by the variational iteration method and the exact solution. For more investigation, we give the values of the approximate and exact solutions of the problem for some points in Table 1. As the numerical results in this table show, the proposed method is effective.

Example 2. In this example we consider the initial value problem

\[ y^{(5)} - 32y = \cos t - 32 \sin t, \]

\[ y(0) = 1, \quad y'(0) = 3, \quad y''(0) = 4, \quad y'''(0) = 7, \quad y^{(4)}(0) = 16. \]

The exact solution of this problem is \( y(t) = e^{2t} + \sin t \). This problem can be easily converted to a system of ODEs as (1) in which

\[ A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 32 & 0 & 0 & 0 & 0 \end{pmatrix}. \]
and $F(t) = (0, 0, 0, \cos t - 32 \sin t)^T$. The approximate solution obtained by the proposed method is

$$\tilde{y} = -31 + 2t + 18t^2 + \frac{4}{3}t^3 - \frac{2}{3}t^4 + \frac{4}{15}t^5 + \frac{2}{15}t^6 + \sin(t) + 32 \cos(t).$$

The exact and approximate solutions are plotted in Fig. 2. As the figure shows, the method gives a very good approximation of the exact solution. We give the values of the approximate and exact solutions of the problem for some points in Table 2. As the numerical results in this table show, the proposed method is effective.

**Example 3.** In this example we give the numerical results of a large problem. Let $n = 50$,

$$A = \begin{pmatrix} -2 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -2 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & -2 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & -2 \end{pmatrix},$$
Table 2
Results for Example 2

<table>
<thead>
<tr>
<th>( t_i )</th>
<th>Approximate solution</th>
<th>Exact solution</th>
<th>Absolute error</th>
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Table 3
Results for Example 3

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<th>Absolute error</th>
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</table>

Fig. 3. Depiction of the approximate and exact solutions of Example 3.

\[ F(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T \] where \( f_i(t) = f_i(t) = 2e^t \) and \( f_i(t) = e^t, i = 2, \ldots, n - 1 \) and \( Y(0) = (1, 1, \ldots, 1)^T \).

The exact solution of this problem is \( y_i(t) = e^t, i = 1, 2, \ldots, n \). These kinds of problems arise in many areas of science and engineering such as discretization of PDEs. By the proposed method this problem was solved. To show the efficiency of the method we give the results of \( y_{10}(t) \). Since, the expression of the approximate solution is large we do not give it here.

The graphs of the approximate solution and exact solutions are displayed in Fig. 3. This figure shows the efficiency of the method. The value of the approximate and exact solutions of the problem for some points are given in Table 3.

5. Conclusion

We have successfully used the variational iteration method for solving a system of ordinary differential equations with constant coefficients. A theorem for the convergence of the method has been given. Results obtained by the method confirm the robustness and efficiency of the method. The method can also be implemented easily.
Acknowledgements

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References