

A Newton two-stage waveform relaxation method for solving systems of nonlinear algebraic equations*

DAVOD KHOJASTEH SALKUYEH^{1,†} AND ZEINAB HASSANZADEH¹

¹ Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran

Received ; accepted ??

Abstract. In this paper, a Newton two-stage waveform relaxation method is introduced to solve systems of nonlinear algebraic equations. The proposed method is derived from the Newton waveform relaxation method by adding further splitting function and inner iterations. Sufficient conditions for the convergence of the method have been provided. Some numerical examples are given to show the effectiveness of the presented method and to compare with two available methods.

AMS subject classifications: 34A09, 34A12.

Key words: Two-stage, Waveform relaxation method, Newton method, Nonlinear algebraic equations, Splitting function, Inner/outer iterations.

1. Introduction

Waveform relaxation (WR) iterative method was first introduced by Lelarasme in his Ph.D. dissertation on the time domain analysis of large scale nonlinear dynamical systems in 1982 [10]. After that, WR method have been investigated in order to numerically solve dynamical linear and nonlinear systems of ordinary differential equations (ODEs), partial differential equations (PDEs) and differential-algebraic equations (DAEs); see for example [3, 7, 8, 11] and references therein. An interesting feature of WR method is the fact that it can be implemented on parallel computations. In [5], WR method was coupled with two-stage iterations strategy for solving initial value problems (IVPs) of ODEs.

Newton method is one of the basic and important numerical method for solving systems of nonlinear equations, which converges locally and quadratically (see [2, 4, 9]). In [12], the Newton method in conjunction with the WR method, which we review it here, has been used to solve system of nonlinear equations. Assume that we are solving the system of equations

$$f(x) = 0, \tag{1}$$

with an initial guess x^0 of the unknown solution x^* where $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$. The key idea of the Newton waveform relaxation method (NWR) for solving system (1) is to choose a splitting function $F : D \times D \rightarrow \mathbb{R}^n$, such that

$$F(x, x) = f(x), \tag{2}$$

*This work was supported by the ...

†Corresponding author. *Email addresses:* khojasteh@guilan.ac.ir, salkuyeh@gmail.com (D.K. Salkuyeh), hassanzadeh.83@gmail.com (Z. Hassanzadeh)

for any $x \in \mathbb{R}^n$, which is called consistency condition. The consistency condition (2) is useful to prove the convergence of NWR iterative method. Denoting the k th iterate by x^k , this then leads to the following iteration scheme

$$F(x^k, x^{k+1}) = 0. \quad (3)$$

The consistency condition (2) ensures that the solution to (1) is a fixed point of (3). In [12], by utilizing the Newton method a procedure of the following form has been stated to solve Eq. (1).

Algorithm 1:

1. **for** $k = 0, 1, \dots$
2. with a given initial approximation \bar{x}^0 of x^{k+1}
3. **for** $m = 0, 1, \dots, M - 1$
4. Solve $F_2(x^k, \bar{x}^m)\Delta\bar{x}^m = -F(x^k, \bar{x}^m)$ for $\Delta\bar{x}^m$
5. $\bar{x}^{m+1} := \bar{x}^m + \Delta\bar{x}^m$
6. **end for**
7. $x^{k+1} := \bar{x}^M$
8. **end for**

In the above procedure $F_2(x^k, \bar{x}^m)$ is the Fréchet derivative [2] of $F(x, y)$ with respect to y at (x^k, \bar{x}^m) , i.e.,

$$F_2(x^k, \bar{x}^m) = \frac{\partial F(x, y)}{\partial y} \Big|_{(x^k, \bar{x}^m)}, \quad (4)$$

which is the $n \times n$ Jacobian matrix evaluated at (x^k, \bar{x}^m) . Moreover, it is assumed that the matrix $F_2(x^k, \bar{x}^m)$ is nonsingular for all $m = 0, 1, \dots, M - 1$, and $k = 0, 1, \dots$. By setting $\bar{x}^0 = x^k$ and $M = 1$ in the above method we obtain the iterative method

$$\begin{cases} F_2(x^k, x^k)\Delta x^k = -f(x^k), \\ x^{k+1} = x^k + \Delta x^k, \end{cases} \quad k = 0, 1, \dots, \quad (5)$$

which is called the *Newton waveform relaxation (NWR)* method. The importance of the splitting function F is that special choice of F can make the matrix $F_2(x, x)$ diagonal and nonsingular. Therefore, NWR method (5) can be processed stably by computer in parallel computations with multi-processor.

In this paper, we establish the *Newton two-stage waveform relaxation (NTSWR)* method for solving system (1), which is generated by selecting a splitting for the function F and adding another nested loop within the inner iteration of Algorithm 1. The advantage of this procedure is its rapid convergence, which is studied in the following sections.

The rest of the paper is organized as follows. After introducing the NTSWR method in Section 2, we investigate the convergence analysis of the method in section 3. Finally, some numerical examples are given in Section 4.

2. Newton two-stage waveform relaxation method

As we mentioned in the previous section, to solve system (1), NWR method is generated in the form (5). We choose the splitting function $G : D \times D \times D \rightarrow \mathbb{R}^n$ for F such that

$$G(x, y, y) = F(x, y), \quad G(x, x, x) = F(x, x) = f(x), \quad (6)$$

for any $x, y \in \mathbb{R}^n$, which is called the consistency condition for the splitting function G . By replacing G in (3) the inner iterations are generated. Hence we have

$$G(x^k, z^v, z^{v+1}) = 0, \quad (7)$$

where v denotes the inner iterations for k th outer iteration. This procedure can be summarized as Algorithm 2.

Algorithm 2:

1. **for** $k = 0, 1, \dots$
2. with a given initial approximation \bar{x}^0 of x^{k+1}
3. $z^0 := x^k$
4. **for** $v = 0, 1, \dots, s - 1$
5. **for** $m = 0, 1, \dots, M - 1$
6. Solve $G_3(x^k, z^v, \bar{z}^m)\Delta\bar{z}^m = -G(x^k, z^v, \bar{z}^m)$ for $\Delta\bar{z}^m$
7. $\bar{z}^{m+1} := \bar{z}^m + \Delta\bar{z}^m$
8. **end for**
9. $z^{v+1} := \bar{z}^M$
10. **end for**
11. $x^{k+1} := z^s$
12. **end for**

In Algorithm 2, $G_3(x^k, z^v, \bar{z}^m)$ is the Fréchet derivative [2] of $G(x, y, z)$ with respect to z at (x^k, z^v, \bar{z}^m) , i.e.,

$$G_3(x^k, z^v, z^{v+1}) = \left. \frac{\partial G(x, y, z)}{\partial z} \right|_{(x^k, z^v, z^{v+1})}. \quad (8)$$

which is the $n \times n$ Jacobian matrix evaluated at (x^k, z^v, \bar{z}^m) .

Similar to the NWR method, by setting $\bar{z}^0 = z^v$, $M = 1$ and using the consistency condition (6) we obtain the following iterative method

$$\begin{cases} G_3(x^k, z^v, z^v)\Delta z^v = -F(x^k, z^v), & z^{v+1} = z^v + \Delta z^v, \\ z^0 = x^k, & x^{k+1} = z^s, \quad k = 0, 1, \dots, \quad v = 0, 1, \dots, s - 1, \\ \text{with a given initial approximation } x^0 \text{ of } x^{k+1}. \end{cases} \quad (9)$$

Hereafter, we call this procedure NTSWR method. In the next section, we give some sufficient conditions for the convergence of this method.

3. Convergence analysis

At first, we recall theorems concerning the convergence of the NWR method which is needful in the next section.

Theorem 1. ([12]) *Let $F : D \times D \rightarrow \mathbb{R}^n$ be a continuously mapping satisfying the consistency condition (2) with $D \subseteq \mathbb{R}^n$ open and convex. In addition, assume that the function $F(x, y)$ is Fréchet differentiable with the second variable y and matrix $F_2(x, y)$ is nonsingular for any $x, y \in D$. If*

$$\|F_2^{-1}(x, x)(F(y, y) - F(x, y))\| \leq \alpha \|x - y\| \quad \forall x, y \in D, \quad (10)$$

$$\|F_2^{-1}(x, x)(F_2(x, x) - F_2(x, y))\| \leq \beta \|x - y\| \quad \forall x, y \in D, \quad (11)$$

$$\begin{aligned} \|x^0 - x^*\| &= \epsilon_0, \quad \varrho = \alpha + \frac{1}{2}\beta\epsilon_0 < 1, \\ B(x^*, r) &\subset D, \quad r = \|x^0 - x^*\|, \end{aligned} \quad (12)$$

where $B(x^*, r)$ is a ball around x^* with radius r , then the sequence $\{x^k\}$ obtained from the NWR method is well defined, remains in the open ball $B(x^*, r)$ and converges to x^* with $F(x^*, x^*) = 0$ (i.e., $f(x^*) = 0$).

Corollary 1. ([12]) *If the conditions concerning the splitting function F in Lemma 1 are fulfilled, and moreover $\alpha < 1$ in (10) and $\beta = 0$ in (11), i.e., the Jacobian matrix $F_2(x, y)$ is independent of the second variable y , then the NWR method converges globally.*

By exerting extensive affine covariant Lipschitz condition [6] for functions of three variables we prove the convergence of the NTSWR method in the following theorems.

Theorem 2. *Let $G : D \times D \times D \rightarrow \mathbb{R}^n$ be a continuously function with D an open and convex set in \mathbb{R}^n satisfying the consistency condition (6). Let also the function $G(x, y, z)$ is Fréchet differentiable with respect to the third variable z and the matrix $G_3(x, y, z)$ is nonsingular for any $x, y, z \in D$. Assume that*

$$\|G_3^{-1}(y, x, x)(G(y, z, z) - G(z, z, z))\| \leq \alpha \|y - z\| \quad \forall x, y, z \in D, \quad (13)$$

$$\|G_3^{-1}(x, y, y)(G(x, y, z) - G(x, z, z))\| \leq \beta \|y - z\| \quad \forall x, y, z \in D, \quad (14)$$

$$\|G_3^{-1}(x, y, y)(G_3(x, y, z) - G_3(x, y, y))\| \leq \gamma \|y - z\| \quad \forall x, y, z \in D, \quad (15)$$

and

$$L = \alpha + \beta + \frac{1}{2}\gamma\epsilon^0 < 1, \quad \epsilon^1 \leq L\epsilon^0 \quad \text{and} \quad B(x^*, \epsilon^0) \subset D \quad (16)$$

where $\|x^j - x^*\| = \epsilon^j$ for $j = 0, 1$, and $B(x^*, \epsilon^0)$ is a ball about x^* with radius ϵ^0 . Then the sequence $\{x^k\}$ obtained from the NTSWR method is well defined, remains in the open ball $B(x^*, \epsilon^0)$ and converges to x^* with $G(x^*, x^*, x^*) = 0$ (i.e., $f(x^*) = 0$).

Proof. Set $e^v = z^v - x^*$ and $e^k = x^k - x^*$. From Eq. (8), $f(x^*) = 0$ and consistency condition (6), we have

$$\begin{aligned} \|e^v + \Delta z^v\| &= \|e^v + G_3^{-1}(x^k, z^v, z^v)(-G(x^k, z^v, z^v)) + G_3^{-1}(x^k, z^v, z^v)f(x^*)\| \\ &= \|G_3^{-1}(x^k, z^v, z^v) (G(x^*, x^*, x^*) - G(x^k, z^v, z^v) + G_3(x^k, z^v, z^v)e^v)\| \\ &\leq \|G_3^{-1}(x^k, z^v, z^v) (G(x^*, x^*, x^*) - G(x^k, x^*, x^*))\| \\ &\quad + \|G_3^{-1}(x^k, z^v, z^v) (G(x^k, x^*, x^*) - G(x^k, z^v, z^v) + G_3(x^k, z^v, z^v)e^v)\|. \end{aligned}$$

By applying condition (13) we derive

$$\begin{aligned} \|e^v + \Delta z^v\| &\leq \|G_3^{-1}(x^k, z^v, z^v) (G(x^k, x^*, x^*) - G(x^k, z^v, z^v) + G_3(x^k, z^v, z^v)e^v)\| \\ &\quad + \alpha \|e^k\| \\ &\leq \alpha \|e^k\| + \|G_3^{-1}(x^k, z^v, z^v) (G(x^k, x^*, x^*) - G(x^k, z^v, x^*))\| \\ &\quad + \|G_3^{-1}(x^k, z^v, z^v) (G(x^k, z^v, x^*) - G(x^k, z^v, z^v) + G_3(x^k, z^v, z^v)e^v)\|. \end{aligned}$$

From the condition (14) we can write

$$\begin{aligned} \|e^v + \Delta z^v\| &\leq \|G_3^{-1}(x^k, z^v, z^v) (G(x^k, z^v, x^*) - G(x^k, z^v, z^v) + G_3(x^k, z^v, z^v)e^v)\| \\ &\quad + \alpha \|e^k\| + \beta \|e^v\| \\ &\leq \|G_3^{-1}(x^k, z^v, z^v) \left(\int_0^1 (G_3(x^k, z^v, z^v) - G_3(x^k, z^v, x^* + se^v)) e^v ds \right)\| \\ &\quad + \alpha \|e^k\| + \beta \|e^v\|. \end{aligned}$$

By putting the condition (15) it is deduced that

$$\|e^v + \Delta z^v\| \leq \alpha \|e^k\| + \beta \|e^v\| + \int_0^1 \gamma \|z^v - x^* - se^v\| \|e^v\| ds.$$

Therefore

$$\|e^v + \Delta z^v\| \leq \alpha \|e^k\| + \|e^v\| \left(\beta + \frac{\gamma}{2} \|e^v\| \right). \quad (17)$$

As we mentioned for a fixed number s of inner iterations, (17) is equivalent to

$$\|z^s - x^* + \Delta z^s\| \leq \alpha \|e^k\| + \|z^s - x^*\| \left(\beta + \frac{\gamma}{2} \|z^s - x^*\| \right). \quad (18)$$

By substituting $z^s = x^{k+1}$, (18) becomes

$$\|e^{k+2}\| \leq \alpha \|e^k\| + \|e^{k+1}\| \left(\beta + \frac{\gamma}{2} \|e^{k+1}\| \right). \quad (19)$$

For the brevity of notation, let $\epsilon^k = \|e^k\|$, $k = 0, 1, \dots$. Then Eq. (19) takes the following form

$$\epsilon^{k+2} \leq \epsilon^{k+1} \left(\beta + \frac{\gamma}{2} \epsilon^{k+1} \right) + \alpha \epsilon^k. \quad (20)$$

By the mathematical principle of induction we show that

$$\begin{cases} \epsilon^{2m-1} \leq L^m \epsilon^0, \\ \epsilon^{2m} \leq L^m \epsilon^0, \end{cases} \quad m = 1, 2, \dots \quad (21)$$

For $m = 1$, we should prove

$$\begin{cases} \epsilon^1 \leq L \epsilon^0, \\ \epsilon^2 \leq L \epsilon^0. \end{cases}$$

The first inequality follows from (16), and to prove the second inequality, invoking Eq. (20) yields

$$\begin{aligned} \epsilon^2 &\leq \epsilon^1 \left(\beta + \frac{\gamma}{2} \epsilon^1 \right) + \alpha \epsilon^0 \\ &\leq L \epsilon^0 \left(\beta + \frac{\gamma}{2} L \epsilon^0 \right) + \alpha \epsilon^0 \\ &\leq \epsilon^0 \left(L \left(\beta + \frac{\gamma}{2} L \epsilon^0 \right) + \alpha \right) \\ &\leq \epsilon^0 \left(\beta + \frac{\gamma}{2} \epsilon^0 + \alpha \right) \quad (\text{from } L < 1) \\ &= L \epsilon^0. \end{aligned}$$

Now assume that Eq. (21) holds for fixed m . Then, for $m + 1$ we have

$$\begin{aligned} \epsilon^{2(m+1)-1} = \epsilon^{2m+1} &\leq \epsilon^{2m} \left(\beta + \frac{\gamma}{2} \epsilon^{2m} \right) + \alpha \epsilon^{2m-1} \\ &\leq L^m \epsilon^0 \left(\beta + \frac{\gamma}{2} \epsilon^0 \right) + \alpha L^m \epsilon^0 \quad (\text{from (21) and } \epsilon^{2m} \leq L^m \epsilon^0 \leq \epsilon^0) \\ &= L^{m+1} \epsilon^0, \end{aligned} \quad (22)$$

and

$$\begin{aligned} \epsilon^{2(m+1)} &= \epsilon^{2m+2} \\ &\leq \epsilon^{2m+1} \left(\beta + \frac{\gamma}{2} \epsilon^{2m+1} \right) + \alpha \epsilon^{2m} \\ &\leq L^{m+1} \epsilon^0 \left(\beta + \frac{\gamma}{2} \epsilon^0 \right) + \alpha L^m \epsilon^0 \quad (\text{from (21), (22) and } \epsilon^{2m+1} \leq L^{m+1} \epsilon^0 \leq \epsilon^0) \\ &\leq L^m \epsilon^0 \left(L \left(\beta + \frac{\gamma}{2} \epsilon^0 \right) + \alpha \right) \\ &= L^{m+1} \epsilon^0, \end{aligned}$$

which completes the induction.

Now, since $0 \leq L < 1$, from (21) we see that

$$\lim_{k \rightarrow +\infty} x^k = x^*,$$

which completes the proof. \square

Corollary 2. *Assume that the conditions concerning the splitting function G in Theorem 2 are fulfilled and moreover $\alpha + \beta < 1$, $\epsilon^1 \leq L\epsilon^0$ and the Jacobian matrix $G_3(x, y, z)$ is independent of the third variable z , i.e., $\gamma = 0$. Then the NTSWR method converges to the solution.*

Proof. Obviously the proof is deduced from Theorem 2. \square

4. Numerical results

In this section, we give three numerical examples to validate the theoretical results and to compare numerically the Newton, NWR and NTSWR methods. All the numerical experiments were computed in double precision using some MATLAB codes on a Pentium 4 PC, with a 3.00 GHz CPU and 1.96GB of RAM. In general, for an arbitrary given system of nonlinear equations, it is time consuming and difficult to verify the sufficient conditions for the convergence given in theorems of the previous section and well as the convergence conditions of the Newton method given in the literature. However, to illustrate the presented theoretical results we verify the sufficient conditions for the convergence of the NWR and NTSWR methods for Example 1.

Example 1. Consider the system of nonlinear equations

$$f(x) = \begin{pmatrix} 2 \sin x_1 + e^{\sin x_2} \\ x_1 + \sin x_2 + 3 \end{pmatrix} = 0. \quad (23)$$

We solve this example by the Newton, NWR and NTSWR methods. Let

$$D = [-3, 0] \times [-0.3, -0.2].$$

The splitting function F is supposed to be

$$F(x^k, x^{k+1}) = \begin{pmatrix} 2 \sin x_1^k + (a(x_2^{k+1} - x_2^k) + 1)e^{\sin x_2^k} \\ ax_1^{k+1} + (1-a)x_1^k + \sin x_2^k + 3 \end{pmatrix}. \quad (24)$$

It is clear that $F(x, x) = f(x)$, therefore the consistency condition (2) is satisfied for any $0 \neq a \in \mathbb{R}$. We consider the splitting function G for F in the following form

$$G(x^k, z^v, z^{v+1}) = \begin{pmatrix} 2 \sin x_1^k + (ab(z_2^{v+1} - z_2^v) + az_2^v - ax_2^k + 1)e^{\sin x_2^k} \\ ab(z_1^{v+1} - z_1^v) + az_1^v + (1-a)x_1^k + \sin x_2^k + 3 \end{pmatrix}. \quad (25)$$

Obviously, we have $G(x, z, z) = F(x, z)$ and $G(x, x, x) = F(x, x) = f(x)$, and hence G satisfies the consistency conditions (6) for any $a, b \in \mathbb{R}$ and $a, b \neq 0$. We first investigate the conditions of Corollary 1 to apply the NWR method. According to the definition of the Jacobian matrix, we have

$$F_2(x^k, x) = \begin{pmatrix} 0 & ae^{\sin x_2^k} \\ a & 0 \end{pmatrix}. \quad (26)$$

Thus the matrix function $F_2(x^k, x)$ is nonsingular for any $x \in \mathbb{R}^2$ and $a \neq 0$, and the Lipschitz condition (11) holds with the constant $\beta = 0$. From (24) and (26), we can write

$$\begin{aligned} & \|F_2^{-1}(x^k, x^k)(F(x^{k+1}, x^{k+1}) - F(x^k, x^{k+1}))\|_1 \\ &= \left\| \left(\begin{array}{c} \frac{(1-a)(x_1^{k+1} - x_1^k) + \sin x_2^{k+1} - \sin x_2^k}{a} \\ \frac{e^{\sin x_2^{k+1}} - e^{\sin x_2^k} + 2(\sin x_1^{k+1} - \sin x_1^k) - ae^{\sin x_2^k}(x_2^{k+1} - x_2^k)}{ae^{\sin x_2^k}} \end{array} \right) \right\|_1. \end{aligned} \quad (27)$$

By applying the mean value theorem, we have

$$\begin{cases} \sin x_2^{k+1} - \sin x_2^k = (\cos \gamma_1)(x_2^{k+1} - x_2^k), & \gamma_1 \in (x_2^k, x_2^{k+1}), \\ e^{\sin x_2^{k+1}} - e^{\sin x_2^k} = (\cos \gamma_2)e^{\sin \gamma_2}(x_2^{k+1} - x_2^k), & \gamma_2 \in (x_2^k, x_2^{k+1}), \\ \sin x_1^{k+1} - \sin x_1^k = (\cos \gamma_3)(x_1^{k+1} - x_1^k), & \gamma_3 \in (x_1^k, x_1^{k+1}). \end{cases} \quad (28)$$

Substituting relations (28) in (27) yields

$$\begin{aligned} & \|F_2^{-1}(x^k, x^k)(F(x^{k+1}, x^{k+1}) - F(x^k, x^{k+1}))\|_1 \\ &= \left\| \left(\begin{array}{c} \frac{(1-a)(x_1^{k+1} - x_1^k) + (\cos \gamma_1)(x_2^{k+1} - x_2^k)}{a} \\ \frac{((\cos \gamma_2)e^{\sin \gamma_2} - ae^{\sin x_2^k})(x_2^{k+1} - x_2^k) + 2(\cos \gamma_3)(x_1^{k+1} - x_1^k)}{ae^{\sin x_2^k}} \end{array} \right) \right\|_1 \leq \alpha(a) \|x^{k+1} - x^k\|_1, \end{aligned}$$

where $\alpha(a)$ is given by

$$\begin{aligned} \alpha(a) &= \max \left\{ \left| \frac{1-a}{a} \right|, \max_{\gamma_1} \left| \frac{\cos \gamma_1}{a} \right|, \max_{\gamma_2, x_2^k} \left| \frac{(\cos \gamma_2)e^{\sin \gamma_2} - ae^{\sin x_2^k}}{ae^{\sin x_2^k}} \right|, \max_{\gamma_3, x_2^k} \left| \frac{2 \cos \gamma_3}{ae^{\sin x_2^k}} \right| \right\} \\ &\leq \max \left\{ \left| \frac{1-a}{a} \right|, \frac{1}{|a|}, \max_{\gamma_2, x_2^k} \left| \frac{1}{a} \cos \gamma_2 e^{\sin \gamma_2 - \sin x_2^k} - 1 \right|, \frac{2e}{|a|} \right\} =: P. \end{aligned}$$

We set $a = 6$. In this case it is easy to see that $P \leq 0.91$. Hence the conditions of Corollary 1 is satisfied and therefore the NWR method (5) can be applied.

In continuation, we peruse the conditions of Corollary 2 for splitting function G to use the NTSWR method. Obviously the matrix

$$G_3(x^k, z^v, z) = \begin{pmatrix} 0 & abe^{\sin x_2^k} \\ ab & 0 \end{pmatrix}, \quad (29)$$

is nonsingular for any $z \in \mathbb{R}^2$ and with the Lipschitz constant $\gamma = 0$. Similar to the

NWR method, by (25), (29) and the mean value theorem, we have

$$\begin{aligned}
& \|G_3^{-1}(z^v, x^k, x^k)(G(z^v, z^{v+1}, z^{v+1}) - G(z^{v+1}, z^{v+1}, z^{v+1}))\|_1 \\
&= \left\| \begin{pmatrix} \frac{\sin z_2^{v+1} - \sin z_2^v + (1-a)(z_1^{v+1} - z_1^v)}{ab} \\ \frac{2(\sin z_1^{v+1} - \sin z_1^v) + e^{\sin z_2^{v+1}} - e^{\sin z_2^v} - ae^{\sin z_2^v}(z_2^{v+1} - z_2^v)}{abe^{\sin x_2^k}} \end{pmatrix} \right\|_1 \\
&= \left\| \begin{pmatrix} \frac{\cos \gamma_4(z_2^{v+1} - z_2^v) + (1-a)(z_1^{v+1} - z_1^v)}{ab} \\ \frac{2(\cos \gamma_5)(z_1^{v+1} - z_1^v) + ((\cos \gamma_6)e^{\sin \gamma_6} - ae^{\sin z_2^v})(z_2^{v+1} - z_2^v)}{abe^{\sin x_2^k}} \end{pmatrix} \right\|_1 \\
&\leq \alpha(a, b) \|z^{v+1} - z^v\|_1.
\end{aligned}$$

Therefore

$$\begin{aligned}
\alpha(a, b) &= \max \left\{ \left| \frac{1-a}{ab} \right|, \max_{\gamma_1} \left| \frac{\cos \gamma_4}{ab} \right|, \max_{\gamma_6, z_2^v, x_2^k} \left| \frac{(\cos \gamma_6)e^{\sin \gamma_6} - ae^{\sin z_2^v}}{abe^{\sin x_2^k}} \right|, \max_{\gamma_5, x_2^k} \left| \frac{2 \cos \gamma_5}{abe^{\sin x_2^k}} \right| \right\} \\
&\leq \max \left\{ \left| \frac{1-a}{ab} \right|, \frac{1}{|ab|}, \max_{\gamma_6, z_2^v, x_2^k} \left| \frac{(\cos \gamma_6)e^{\sin \gamma_6} - ae^{\sin z_2^v}}{abe^{\sin x_2^k}} \right|, \frac{2e}{|ab|} \right\} =: Q_1.
\end{aligned}$$

We set $a = 6$ and $b = 1.1$. It can be seen that $Q_1 \leq 0.86$. Similarly,

$$\begin{aligned}
& \|G_3^{-1}(x^k, z^v, z^v)(G(x^k, z^v, z^{v+1}) - G(x^k, z^{v+1}, z^{v+1}))\|_1 \\
&= \left\| \begin{pmatrix} \frac{a(1-b)(z_1^{v+1} - z_1^v)}{ab} \\ \frac{a(1-b)e^{\sin x_2^k}(z_2^{v+1} - z_2^v)}{abe^{\sin x_2^k}} \end{pmatrix} \right\|_1 \leq \beta(a, b) \|z^{v+1} - z^v\|_1.
\end{aligned}$$

Hence

$$\beta(a, b) = \left| \frac{a(1-b)}{ab} \right| = \frac{1}{11} =: Q_2.$$

Therefore, we have

$$\alpha(a, b) + \beta(a, b) \leq Q_1 + Q_2 = 0.96 < 1.$$

Thereupon the NTSWR method (8) can be applied to solve the given system of nonlinear equations. We compare the numerical results of the Newton, NWR and NTSWR methods. To do so, the number of inner iterations is set to be $s = 1$ and the outer iterations is terminated as soon as the stopping criterion $\|f(x^k)\|_\infty \leq 10^{-14}$ holds. We consider $x^0 = (-1, -0.28)^T$ and $x^0 = (1, 1)^T$ as the initial vectors. The number of iterations of the method together with the CPU times (in parenthesis and in seconds) for the convergence of the method are given in Table 1. It is noted that each code is executed ten number times and the minimum time obtained is quoted in the tables. The convergence history of the methods is displayed in Figure 1 and Figure 2. As seen for both of the starting points $x^0 = (-1, -0.28)^T$ and $x^0 = (1, 1)^T$ the Newton method fails to converge. In fact, the Newton method has not converged

in 400 iterations and a stagnation has been occurred. In terms of the number of iterations the NTSWR method is superior to the NWR method, and it is roughly half that of the NWR method. However, the CPU times of the NTSWR method is slightly greater than that of the NWR method. It is necessary to mention that the initial guess $x^0 = (1, 1)^T$ is not in D . Nevertheless, both of the NWR and NTSWR methods are convergent.

Initial guess	Newton	NWR	NTSWR
$x^0 = (-1, -0.28)^T$	Fail	245(0.078)	133(0.109)
$x^0 = (1, 1)^T$	Fail	251(0.078)	134(0.109)

Table 1: Numerical results for Example 1 with $s = 1$.

In continuation, we investigate the effect of number of inner iterations on the outer iterations. To do so, we report the number of outer iterations for different values of s in Table 2. This table shows that for small values of s (e.g., $s = 1, 2$) the NTSWR method provides quite suitable results in comparison with the NWR method. An interesting observation is that the results of the NTSWR method with $s = 1$ is better than those of the NWR method.

Initial guess	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$
$x^0 = (-1, -0.28)^T$	133	128	127	127	127
$x^0 = (1, 1)^T$	134	130	130	130	130

Table 2: Effect of the number of inner iterations on outer iterations for Example 1.

Example 2. Consider the following nonlinear equations

$$f(x) = \begin{pmatrix} 10^9 \arctan(\frac{x_1}{10^9}) + e^{\sin x_2} + 3 \\ x_1 + x_2 - \sin(3x_1) \end{pmatrix} = 0. \quad (30)$$

This system was examined by the NWR method in [12] by using the splitting function

$$F(x^k, x^{k+1}) = \begin{pmatrix} 10^9 \arctan(\frac{x_1^k}{10^9}) + [a(x_1^{k+1} - x_1^k) + 1]e^{\sin x_2^k} + 3 \\ x_1^k + ax_2^{k+1} + (1-a)x_2^k - \sin(3x_1^k) \end{pmatrix}.$$

We use

$$G(x^k, z^v, z^{v+1}) = \begin{pmatrix} 10^9 \arctan(\frac{x_1^k}{10^9}) + [ab(z_1^{v+1} - z_1^v) + az_1^v - ax_1^k + 1]e^{\sin x_2^k} + 3 \\ x_1^k + ab(z_2^{v+1} - z_2^v) + (1-a)x_2^k + az_2^v - \sin(3x_1^k) \end{pmatrix},$$

as the splitting function in the NTSWR method. In this example, the number of inner iterations is set to be $s = 1$ and the stopping criterion $\|f(x^k)\|_\infty \leq 10^{-14}$ is used for the outer iterations. We also set $a = 10.25$ and $b = 0.75$. We have checked two initial vectors $x^0 = (-2, 2)^T$ and $x^0 = (2, -2)^T$ and the corresponding numerical results are given in Table 3. For both of the starting vectors the Newton method

fails to converge. As observed the NTSWR is superior to the NWR in terms of the number of iterations. In fact the number of iterations of the NTSWR has roughly been reduced by a factor of three in both of the cases. On the other hand, the CPU times for both of the methods are the same. The convergence history is displayed in Figures 3 and 4.

Initial guess	Newton	NWR	NTSWR
$x^0 = (-2, 2)^T$	Fail	212(0.078)	77(0.078)
$x^0 = (2, -2)^T$	Fail	218(0.078)	79(0.078)

Table 3: Numerical results for Example 2 with $s = 1$.

Similar to the previous example, we report the number of outer iterations with respect to the inner iteration numbers. As seen the NTSWR method provides quite suitable results for small values of s in comparison with the NWR method.

Initial guess	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$
$x^0 = (-2, 2)^T$	77	94	88	90	90
$x^0 = (2, -2)^T$	79	97	90	92	92

Table 4: Effect of the number of inner iterations on outer iterations for Example 2.

Example 3. Let $f : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ with the following nonlinear equations

$$f(x) = x + \phi(x) = 0, \quad \phi_i(x) = e^{\cos(\sum_{j=1}^i x_j)}, \quad i = 1, \dots, 5. \quad (31)$$

We consider the splitting functions F and G as follows

$$F(x^k, x^{k+1}) = ax^{k+1} + (1-a)x^k + \phi(x^k),$$

and

$$G(x^k, z^v, z^{v+1}) = ab(z^{v+1} - z^v) + az^v + (1-a)x^k + \phi(x^k).$$

We set $a = 14$, $b = 0.5$ and $s = 1$. The stopping criterion for the outer iterations is set to be $\|f(x^k)\|_\infty \leq 10^{-14}$. We use $x^0 = (-5, \dots, -5)^T$ and $x^0 = (2, \dots, 2)^T$ as the initial vectors and the numerical results are given in Table 5. As seen, for both of the initial vectors the Newton method fails to converge, but both of the NWR and NTSWR methods converge properly. Roughly speaking, the number of iterations of the NWR method is four times that of the NTSWR method for both of the initial vectors. Moreover, the CPU times for the NWR method is greater than that of the NTSWR method. The convergence history is displayed in Figures 5 and 6.

Similar to the previous examples, we report the number of outer iterations with respect to the inner iterations in Table 6.

Initial guess	Newton	NWR	NTSWR
$x^0 = (-5, \dots, -5)^T$	Fail	548(0.359)	133(0.218)
$x^0 = (2, \dots, 2)^T$	Fail	428(0.281)	103(0.172)

Table 5: Numerical results for Example 3 with $s = 1$.

Initial guess	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$
$x^0 = (-5, \dots, 5)^T$	133	266	133	266	133
$x^0 = (2, \dots, 2)^T$	103	205	103	205	103

Table 6: Effect of the number of inner iterations on outer iterations for Example 3.

5. Conclusion

In this paper, we have presented Newton two-stage waveform relaxation method to solve systems of nonlinear algebraic equations. Then, we have given some sufficient conditions for the convergence of the method. Next, we have compared numerical results of the proposed method with those of the Newton waveform relaxation method and the classical Newton method. Numerical results show that the proposed method can be considered as an efficient method to solve system of nonlinear equations.

Acknowledgements

The authors would like to thank the anonymous referee for the valuable comments and suggestions which improved the quality of the paper.

References

- [1] E. ALLGOWER, K. GEORG, *Simplicial and continuation methods for approximating fixed points and solutions to systems of equations*, SIAM Rev. **22**(1980), 158.
- [2] K. ATKINSON, W. HAN, *Theoretical Numerical Analysis*, Springer, Iowa U.S.A, 2005.
- [3] W. BAO, Y. SONG, *Two stage waveform relaxation method for the initial value problems of differential-algebraic equations*, J. Comput. Appl. Math. **236**(2011), 1123–1136.
- [4] R. L. BURDEN, J. D. FAIRES, *Numerical Analysis*, Ninth Edition, Youngstown State University, U.S.I, 2011.
- [5] R. GARRAPPA, *An analysis of convergence for two-stage waveform relaxation methods*, J. Comput. Appl. Math. **169**(2004), 377-392.
- [6] R. H. W. HOPPE, *Numerical Methods for Large-Scale Nonlinear Systems*, Springer, Berlin-Heidelberg-New York, 2004.
- [7] K. J. HOUT, *On the convergence of waveform relaxation methods for stiff nonlinear ordinary differential equations*, Appl. Numer. Math. **18**(1995), 175-190.
- [8] J. K. M. JANSSEN, R. M. M. MATTHEIJ, M. T. M. PENDERS, W. H. A. SCHILDERS, *Stability and efficiency of waveform relaxation methods*, Comput. Math. Appl. **28**(1994), 153166.
- [9] R. KRESS, *Numerical Analysis*, Springer-Verlag, New York, 1998.
- [10] E. LELARSMEE, *The Waveform Relaxation Method for the Time Domain Analysis of Large Scale Nonlinear Dynamical Systems*, Ph.D. thesis, University of California, Berkeley, 1982.

- [11] J. WANG, Z. Z. BAI, *Convergence analysis of two-stage waveform relaxation method for the initial value problems*, Appl. Math. Comput. **172**(2006), 797-808.
- [12] S. WU, C. HUANG, Y. LIU, *Newton waveform relaxation method for solving algebraic nonlinear equations*, Appl. Math. Comput. **201**(2008), 553-560.

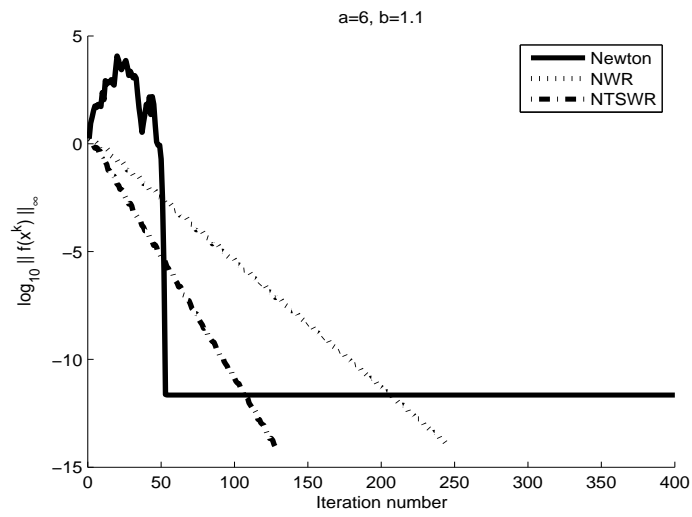


Figure 1: $\log_{10} \|f(x^k)\|_{\infty}$ with initial guess $x^0 = (-1, -0.28)^T$ for Example 1.

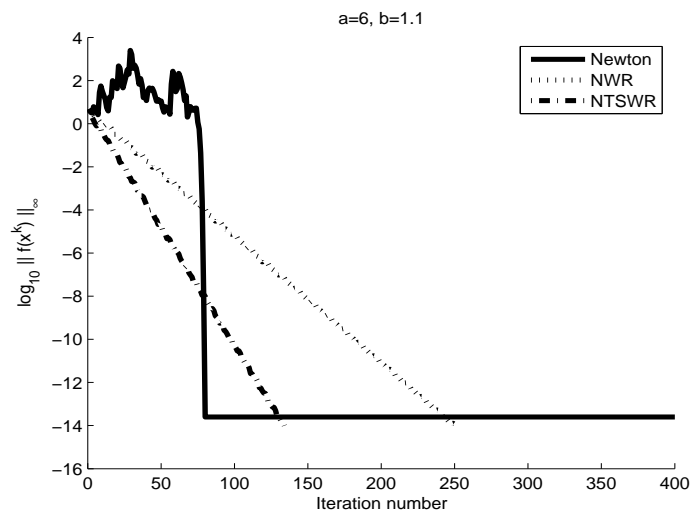


Figure 2: $\log_{10} \|f(x^k)\|_{\infty}$ with initial approximation $x^0 = (1, 1)^T$ for Example 1.

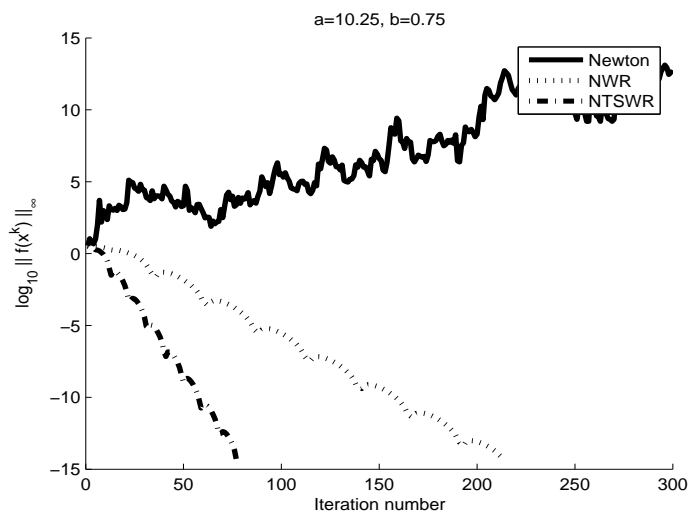


Figure 3: $\log_{10} \|f(x^k)\|_{\infty}$ with initial guess $x^0 = (-2, 2)^T$ for Example 2.

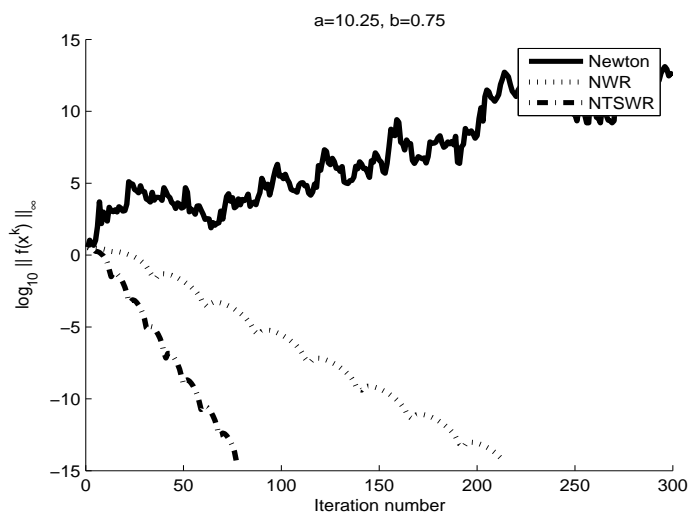


Figure 4: $\log_{10} \|f(x^k)\|_{\infty}$ with initial approximation $x^0 = (2, -2)^T$ for Example 2.

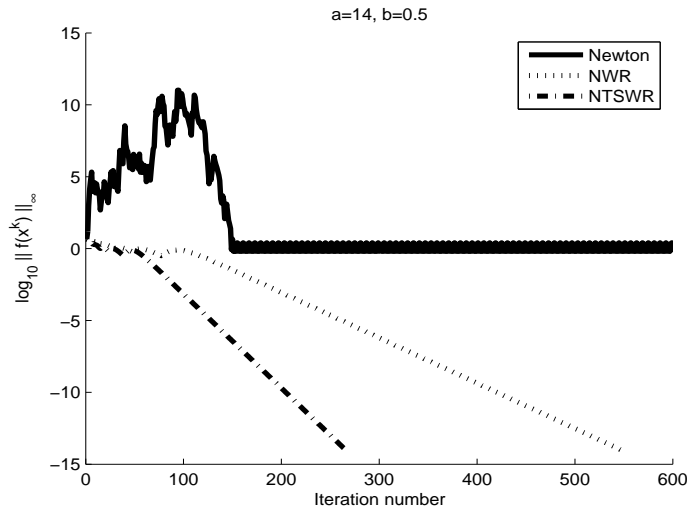


Figure 5: $\log_{10} \|f(x^k)\|_\infty$ with initial guess $x^0 = (-5, \dots, -5)^T$ for Example 3.

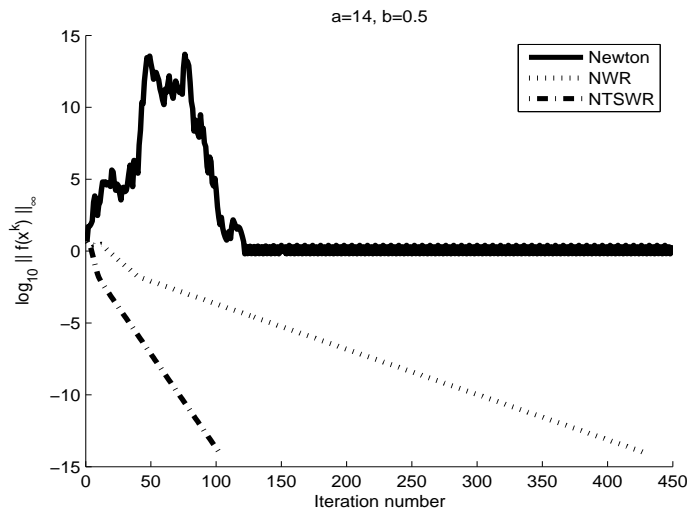


Figure 6: $\log_{10} \|f(x^k)\|_\infty$ with initial approximation $x^0 = (2, \dots, 2)^T$ for Example 3.