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Two-parameter generalized Hermitian and skew-Hermitian splitting iteration method

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It is known that the Hermitian and skew-Hermitian splitting (HSS) iteration method is an efficient solver for non-Hermitian positive definite linear system of equations. In [M. Benzi, SIAM J. Matrix Anal. Appl. 31 (2009) 360–374] Benzi proposed a generalization of the HSS (GHSS) iteration method. In this paper, we present a two-parameter version of the GHSS (TGHSS) method and investigate its convergence properties. To show the effectiveness of the proposed method the TGHSS iteration method is applied to image restoration and convection-diffusion problems and the results are compared with those of the HSS and GHSS methods.

Keywords: two-parameter; HSS; GHSS; image restoration; convection-diffusion problem.

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1. Introduction

Consider the system of linear equations

$$Ax = b, \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$ is a large and sparse non-Hermitian positive definite matrix (i.e. its Hermitian part $A + A^*$ is Hermitian positive definite) and $x, b \in \mathbb{C}^n$. In [17], Golub and Vanderstraeten presented a new preconditioner for the linear system (1) by means of the Hermitian and skew-Hermitian splitting of the matrix A. In fact, they used the shifted version of the Hermitian and skew-Hermitian parts of the coefficient matrix. Bai, Golub and Ng [6] presented the Hermitian and skew-Hermitian splitting (HSS) method to solve the linear system (1). In fact,they used the shifted matrices $\alpha I + H$ and $\alpha I + S$ to present the HSS iteration method where $H = (A + A^*)/2$ and $S = (A - A^*)/2$ are the Hermitian and skew-Hermitian parts of A, respectively. Moreover, they proved that if A is positive definite matrix, then the HSS method converges to the unique solution of the linear system for any positive α.

During recent years, lots of studies have been done to generalize, modify and enhance the performance of the HSS method. The preconditioned HSS (PHSS) method has been used to solve non-Hermitian positive semidefinite linear systems in [9]. The successive-overrelaxation (SOR) acceleration scheme and its convergence results have also been

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investigated in [7]. The lopsided HSS (LHSS) iteration method and its inexact version (ILHSS) has been applied to solve non-Hermitian positive definite linear systems in [27, 35]. A class of complex symmetric linear systems have been solved by the modified HSS (MHSS) method in [3]. The HSS method has also been applied to approximately solve the Sylvester matrix equation $AX + XB = F$ and the linear matrix equation $AXB = C$, respectively, in [1] and [36]. Wang, Li and Mao [37] have split the matrix into positive-definite and skew-Hermitian parts and proposed the positive-definite and skew-Hermitian splitting (PSS) iteration methods to solve the continuous Sylvester matrix equation $AX + XB = C$. Recently, Krukier, Krukier and Ren [25] have used a new splitting for skew-Hermitian part and generalized the skew-Hermitian triangular splitting iteration methods to solve saddle point problem. In [10], Bai and Yang used an HSS-based iteration method to solve weakly nonlinear systems.

Bai and Golub [4] presented the accelerated Hermitian and skew-Hermitian splitting (AHSS) iteration method for saddle-point problems. Chen and Jiang applied the AHSS iteration method to solve nonsymmetric positive definite Toeplitz systems in [15]. The two-parameter (asymmetric) version of the HSS (THSS) method has been presented by Li, Huang and Liu in [26]. They inserted another parameter to the HSS method and investigated its convergence and effectiveness. The THSS method have also been applied to solve singular linear system of equations by Li, Liu and Peng [28]. Yang and Wu [40] generalized the PHSS (GPHSS) method to solve non-Hermitian positive definite linear systems. They used two parameters in the classical PHSS method to increase the efficiency and flexibility of their method. In a similar study, Yin and Dou [41] used a new parameters and preconditioner for each step of the PHSS method. Inserting a new parameter to each step of the iteration has been done for different versions of the HSS method. Wang, Wu and Fan [38] presented a new parameter for each step of the preconditioned normal and skew-Hermitian splitting (NSS) iteration method to solve non-Hermitian and positive definite linear systems.

It is well known that if either H or S dominates in the two-step HSS method, then HSS can be considered as a good preconditioner. Benzi in [11] used the proposed domination concept and presented generalized HSS (GHSS) method by using a new splitting to Hermitian part of the coefficient matrix. In the GHSS method, the Hermitian part of matrix A is split into $H = G + K$, where G and K are Hermitian positive semidefinite matrices. To implement the GHSS method, the matrix K is associated with the skew-Hermitian part (i.e. $A = H + S = G + (K + S)$). Hence, the domination of the S is improved. In fact, in the second step of the GHSS method, a more well-conditioned system is solved than HSS method.

In this study, we use the idea of the THSS and GHSS methods to present the two-parameter GHSS (TGHSS) iteration method. In fact, two different parameters are considered in two half-steps of the GHSS iteration method. These parameters allow us to have more control on the each of the half-steps of the TGHSS. We then investigate the convergence and effectiveness of the TGHSS method.

Throughout the paper, $\langle .,. \rangle$ stands for the standard inner product in $\mathbb{C}^n$, i.e., for $x, y \in \mathbb{C}^n$ $\langle x, y\rangle = x^H y$, and for any vector $x \in \mathbb{C}^n$, $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$. Similarly, for any matrix $A \in \mathbb{C}^{n\times n}$, $\|A\| = \max_{\|x\|=1} \|Ax\|$. For a matrix $A \in \mathbb{C}^{n\times n}$, $\rho(A)$ and $\sigma(A)$ stand for the spectral radius and the spectrum of A, respectively. For a matrix $X \in \mathbb{R}^{n\times n}$ whose eigenvalues are real, its largest and smallest eigenvalues are respectively denoted by $\lambda_1(X)$ and $\lambda_n(X)$.

This paper is organized as follows. In Section 2, after giving a brief description of the HSS and GHSS methods, we present the TGHSS method and investigate its convergence properties. Two numerical examples are given in Section 3 to illustrate the presented theoretical results. To this end, image restoration problem and convection-diffusion problem are investigated. Finally, some concluding remarks are given in Section 4.

2. Description of the method

In this section, we first briefly introduce the HSS and the GHSS methods and then present the TGHSS method.

2.1 *A brief description of the HSS and GHSS methods*

In the HSS method the matrix A is split as $A = H + S$, where H and S are the Hermitian and skew-Hermitian parts of the matrix A, respectively. Consider the following two splittings of A

$$A = (H + \alpha I) - (\alpha I - S) \text{ and } A = (S + \alpha I) - (\alpha I - H), \tag{2}$$

where α is positive parameter and I is the identity matrix of order n. By using the proposed splittings, the alternating iteration for the HSS method is introduced as follows

$$\begin{cases} (H + \alpha I)x_{k+\frac{1}{2}} = (\alpha I - S)x_k + b, \\ (S + \alpha I)x_{k+1} = (\alpha I - H)x_{k+\frac{1}{2}} + b, \end{cases} \qquad k = 0, 1, \dots. \tag{3}$$

After eliminating of $x_{k+\frac{1}{2}}$ from Eq. (3) we obtain the stationary iteration method

$$x_{k+1} = T_\alpha x_k + P_\alpha^{-1} b, \quad k = 0, 1, \dots,$$

where

$$T_\alpha = I - P_\alpha^{-1} A, \quad P_\alpha = \frac{1}{2\alpha}(H + \alpha I)(S + \alpha I).$$

It is noted that the HSS method is a stationary iteration scheme generated based on the splitting $A = P_\alpha - (P_\alpha - A)$ and as a result P_α is the HSS preconditioner.

In [6], it has been shown that for any positive constant α the HSS iteration method unconditionally converges to the exact solution of $Ax = b$ for any initial guess $x^{(0)} \in \mathbb{C}^n$. As Benzi mentioned in [11], the HSS method is especially effective if either H or S dominates. In this case, it can be easily seen that $P_\alpha \to \frac{1}{2}(A + \alpha I)$. Hence, for the small values of α, P_α is a good approximation for A. However, if we consider the Hermitian part of the matrix A as $H = \epsilon L + K$, where $L = L^*$ is a positive definite and $K = K^*$ is positive semidefinite matrices and $\epsilon > 0$ is a (small) parameter, then as the author mentioned in [11], we have

$$\begin{aligned} P_\alpha &= \frac{1}{2\alpha}(\epsilon L + K + \alpha I)(S + \alpha I) \\ &= \frac{1}{2\alpha}(KS + \alpha K + \alpha S) + \frac{\epsilon}{2\alpha}(LS + \alpha L) + \frac{\alpha}{2} I. \end{aligned}$$

Hence, the P_α is not a good approximation for A for fixed value of α and as $\epsilon \to 0$. Therefore, the following splittings for A have been introduced by Benzi [11]

$$A = (\alpha I + G) - (\alpha I - S - K), \quad A = (\alpha I + S + K) - (\alpha I - G), \tag{4}$$

where $G = \epsilon L$ and ϵ is small positive constant. Then, the GHSS iteration method based on the above splittings is stated as

$$\begin{cases} (G+\alpha I)x_{k+\frac{1}{2}} = (\alpha I - S - K)x_k + b, \\ (S+K+\alpha I)x_{k+1} = (\alpha I - G)x_{k+\frac{1}{2}} + b, \end{cases} \qquad k = 0, 1, \ldots. \tag{5}$$

In this case, for small values of α and $\epsilon \to 0$ the GHSS preconditioner satisfies the following relations

$$P_\alpha = \frac{1}{2\alpha}(\epsilon L + \alpha I)(S + K + \alpha I) \tag{6}$$

$$= \frac{1}{2}A + \frac{\epsilon}{2\alpha}L(S+K) + \frac{\alpha}{2}I \to \frac{1}{2}(A + \alpha I). \tag{7}$$

As seen, the GHSS preconditioner P_α is a good approximation for the matrix A. In the next subsection we propose the TGHSS iteration method.

2.2 *Two-parameter GHSS method (TGHSS)*

Consider the two following splittings for the matrix A

$$A = (\alpha I + G) - (\alpha I - S - K), \quad A = (\beta I + S + K) - (\beta I - G),$$

where α and β are positive parameters. Then, we propose the TGHSS method as following

$$\begin{cases} (\alpha I + G)x_{k+\frac{1}{2}} = (\alpha I - S - K)x_k + b, \\ (\beta I + S + K)x_{k+1} = (\beta I - G)x_{k+\frac{1}{2}} + b, \end{cases} \qquad k = 0, 1, \ldots. \tag{8}$$

Similar to the HSS iteration method, TGHSS can be written as

$$x_{k+1} = T_{\alpha,\beta}x_k + P_{\alpha,\beta}^{-1}b \tag{9}$$

where

$$T_{\alpha,\beta} = I - P_{\alpha,\beta}^{-1}A, \qquad P_{\alpha,\beta} = \frac{1}{\alpha+\beta}(\alpha I + G)(\beta I + S + K). \tag{10}$$

Hence, $P_{\alpha,\beta}$ would be the TGHSS preconditioner. We have

$$P_{\alpha,\beta} = \frac{1}{\alpha+\beta}(\alpha I + \epsilon L)(\beta I + S + K), \tag{11}$$

$$= \frac{\alpha}{\alpha+\beta}A + \frac{1}{\alpha+\beta}\epsilon L(S + K + (\beta - \alpha)I) + \frac{\alpha\beta}{\alpha+\beta}I. \tag{12}$$

Now, for fixed α and small values of β (for example $\beta = \sqrt{\epsilon}$), it can be seen that $P_{\alpha,\beta}$ is a good approximation for the matrix A. For example, if $\alpha = 1$ and $\beta = \sqrt{\epsilon}$, then $P_{\alpha,\beta} \to A$ as $\epsilon \to 0$.

In the sequel, the convergence of the TGHSS method is investigated. To this end, we first present a new generalization of the Kellogg's lemma [30]. A similar version of the following lemma can be found in [24].

LEMMA 2.1 *Let $A \in \mathbb{C}^{n\times n}$ and $H = \frac{1}{2}(A + A^*)$ be positive definite. Let also α and β be two nonnegative parameters.*

(i) *If $0 \le \alpha \le \beta + \lambda_n/2$, then $\|(\alpha I - A)(\beta I + A)^{-1}\| \le 1$;*

(ii) *If $0 \le \alpha < \beta + \lambda_n/2$ such that α and β are not simultaneously equal to zero, then $\|(\alpha I - A)(\beta I + A)^{-1}\| < 1$,*

where λ_n is the smallest eigenvalue of H.

Proof. Let $T = (\alpha I - A)(\beta I + A)^{-1}$ and $\psi = (\beta I + A)^{-1}\phi$. Then, we have

$$\begin{aligned}
\|T\|^2 &= \sup_{\phi\neq 0} \frac{\langle T\phi, T\phi\rangle}{\langle \phi, \phi\rangle} \\
&= \sup_{\psi\neq 0} \frac{\langle (\alpha I - A)\psi, (\alpha I - A)\psi\rangle}{\langle (\beta I + A)\psi, (\beta I + A)\psi\rangle} \\
&= \sup_{\psi\neq 0} \frac{\alpha^2\langle \psi, \psi\rangle - \alpha\langle \psi, A\psi\rangle - \alpha\langle A\psi, \psi\rangle + \langle A\psi, A\psi\rangle}{\beta^2\langle \psi, \psi\rangle + \beta\langle \psi, A\psi\rangle + \beta\langle A\psi, \psi\rangle + \langle A\psi, A\psi\rangle} \\
&= \sup_{\psi\neq 0} \frac{\alpha^2\langle \psi, \psi\rangle - \alpha\langle (A + A^H)\psi, \psi\rangle + \langle A\psi, A\psi\rangle}{\beta^2\langle \psi, \psi\rangle + \beta\langle (A + A^H)\psi, \psi\rangle + \langle A\psi, A\psi\rangle} \\
&= \sup_{\psi\neq 0} \frac{\alpha^2\langle \psi, \psi\rangle - \frac{\alpha}{2}\langle H\psi, \psi\rangle + \langle A\psi, A\psi\rangle}{\beta^2\langle \psi, \psi\rangle + \frac{\beta}{2}\langle H\psi, \psi\rangle + \langle A\psi, A\psi\rangle}.
\end{aligned}$$

To prove (i), it is enough to show that under the given assumption we have $\|T\| \le 1$. From the above equation it is easy to see that

$$\frac{\alpha^2\langle \psi, \psi\rangle - \frac{\alpha}{2}\langle H\psi, \psi\rangle + \langle A\psi, A\psi\rangle}{\beta^2\langle \psi, \psi\rangle + \frac{\beta}{2}\langle H\psi, \psi\rangle + \langle A\psi, A\psi\rangle} \le 1, \qquad \text{for any } \psi \neq 0, \tag{13}$$

if and only if

$$(\alpha^2 - \beta^2)\|\psi\|^2 - \frac{\alpha + \beta}{2}\langle H\psi, \psi\rangle \le 0, \qquad \text{for any } \psi \neq 0. \tag{14}$$

If $\alpha = \beta = 0$, then there is nothing to prove. Otherwise, Eq. (14) is equivalent to

$$2(\alpha - \beta) \le \frac{\langle H\psi, \psi\rangle}{\langle \psi, \psi\rangle}, \qquad \text{for any } \psi \neq 0. \tag{15}$$

Hence, if

$$2(\alpha - \beta) \le \min_{\psi\neq 0} \frac{\langle H\psi, \psi\rangle}{\langle \psi, \psi\rangle} = \lambda_n, \tag{16}$$

then (15) holds and as a result we would have $\|T\| \le 1$. This completes the proof of (i).

To prove (ii) it is enough to change the inequality sign in Eqs. (13)-(16) to strict inequality. ∎

In Lemma 2.1, when H is positive semidefinite, we have $\lambda_n = 0$. In this case, we can pose the following remark.

Remark 1 Let $A \in \mathbb{C}^{n\times n}$ and $H = \frac{1}{2}(A + A^*)$ be positive semidefinite. Let also α and β be two nonnegative parameters.

(i) If $0 \le \alpha \le \beta$, then $\|(\alpha I - A)(\beta I + A)^{-1}\| \le 1$;

(ii) If $0 \le \alpha < \beta$, then $\|(\alpha I - A)(\beta I + A)^{-1}\| < 1$.

In the next theorem, sufficient conditions for the convergence of the TGHSS iteration are given.

Theorem 2.2 *Let α and β be two nonnegative numbers such that α and β are not simultaneously equal to zero. Let $A = (G + K) + S = H + S$, where G and K are Hermitian positive semidefinite matrices and S is a skew-Hermitian matrix. Alternating iteration (8) converges to the unique solution of $Ax = b$ if one of the following conditions holds:*

(i) *G is positive definite, K is positive semidefinite matrix and*

$$\alpha < \beta \le \alpha + 2\lambda_n(G) \quad \text{or} \quad \alpha \le \beta < \alpha + 2\lambda_n(G).$$

(ii) *G is positive semidefinite, K is positive definite matrix and*

$$\beta \le \alpha < \beta + \frac{1}{2}\lambda_n(K) \quad \text{or} \quad \beta < \alpha \le \beta + \frac{1}{2}\lambda_n(K).$$

(iii) *G is positive definite, K is positive definite matrix and*

$$\begin{aligned} &\alpha < \beta + \frac{1}{2}\lambda_n(K) \le \alpha + 2\lambda_n(G) + \frac{1}{2}\lambda_n(K) \\ \text{or} \quad &\alpha \le \beta + \frac{1}{2}\lambda_n(K) < \alpha + 2\lambda_n(G) + \frac{1}{2}\lambda_n(K). \end{aligned}$$

Proof. From Eq. (10), it can be seen that

$$T_{\alpha,\beta} = I - P_{\alpha,\beta}^{-1}A = (\beta I + S + K)^{-1}(\beta I - G)(\alpha I + G)^{-1}(\alpha I - S - K).$$

We now define the matrix $\hat{T}_{\alpha,\beta}$ as following

$$\begin{aligned} \hat{T}_{\alpha,\beta} &= (\beta I + S + K)T_{\alpha,\beta}(\beta I + S + K)^{-1} \\ &= (\beta I - G)(\alpha I + G)^{-1}(\alpha I - S - K)(\beta I + S + K)^{-1}. \end{aligned}$$

Since $\hat{T}_{\alpha,\beta}$ is similar to $T_{\alpha,\beta}$, to investigate the convergence of the TGHSS method we consider the eigenvalues distribution of the matrix $\hat{T}_{\alpha,\beta}$. From the latter equation it follows that

$$\rho(T_{\alpha,\beta}) = \rho(\hat{T}_{\alpha,\beta}) \le \|T_1\|\|T_2\|, \tag{17}$$

where

$$T_1 = (\beta I - G)(\alpha I + G)^{-1}, \quad T_2 = (\alpha I - S - K)(\beta I + S + K)^{-1}.$$

We prove only case (i) and the other two cases can be proved similarly. Hence, assume that G and K are positive definite and positive semidefinite matrices, respectively. Then,

$$\|T_1\| = \|(\beta I - G)(\alpha I + G)^{-1}\| = \max_{\lambda \in \sigma(G)} \frac{|\beta - \lambda|}{\alpha + \lambda}.$$

Therefore, there exists a $\hat{\beta} > 0$ such that

$$\|T_1\| = \begin{cases} \frac{\lambda_1(G) - \beta}{\lambda_1(G) + \alpha}, & 0 < \beta \leq \hat{\beta}, \\ \frac{\beta - \lambda_n(G)}{\lambda_n(G) + \alpha}, & \hat{\beta} \leq \beta. \end{cases} \tag{18}$$

From the above relation, it is easy to see that if

$$\beta \leq \alpha + 2\lambda_n(G) \tag{19}$$

then $\|T_1\| \leq 1$. On the other hand, if $M = S + K$, then $T_2 = (\alpha I - M)(\beta I + M)^{-1}$. We have

$$\frac{1}{2}(M + M^*) = \frac{1}{2}(S + K + K^* + S^*) = \frac{1}{2}(K + K^*) = K,$$

which is a symmetric positive semidefinite matrix. Therefore, from the second part of Remark 1, if $\alpha < \beta$ then $\|T_2\| < 1$. By considering this and Eq. (19) we see that if $\alpha < \beta \leq \alpha + 2\lambda_n(G)$ then $\rho(T_{\alpha,\beta}) = \rho(\hat{T}_{\alpha,\beta}) < 1$, and the convergence of the TGHSS method is achieved. Furthermore, a similar argument can be used to show that if

$$\beta < \alpha + 2\lambda_n(G), \tag{20}$$

then $\alpha \leq \beta$ is a sufficient condition for the convergence of the proposed method. ■

As mentioned in proof of Theorem 2.2, we have

$$\rho(T_{\alpha,\beta}) \leq \|T_1\|\|T_2\| \leq \max_{\lambda \in \sigma(G)} \frac{|\beta - \lambda|}{\alpha + \lambda} = \sigma(\alpha, \beta),$$

where $\sigma(\alpha, \beta)$ is an upper bound for spectral radius of the matrix $T_{\alpha,\beta}$ when main assumptions of the proposed theorem are fulfilled. Therefore, the convergence speed of the TGHSS is related to $\sigma(\alpha, \beta)$. Indeed, the smaller of the presented bound is, the smaller spectral radius is obtained. Now, we want to find the values of α and β that minimize this upper bound. Since, finding the optimal values of the parameters α and β is a critical task, we compute the optimal value of one of them when the other one is set to be a prescribed value. By the next corollary, for a fixed α (β) in the convergence region, the best (optimal) value for β (α) can be obtained.

Corollary 2.3 *Let α and β be two nonnegative numbers such that α and β are not simultaneously equal to zero. Let $A = (G+K)+S = H+S$, where G and K are Hermitian positive semidefinite matrices and S is a skew-Hermitian matrix. Moreover, let*

$$\tilde{\beta}(\alpha) = \frac{\alpha(\lambda_n(G) + \lambda_1(G)) + 2\lambda_n(G)\lambda_1(G)}{2\alpha + \lambda_n(G) + \lambda_1(G)},$$

and

$$\tilde{\sigma}(\alpha) = \frac{\lambda_1^2(G) + \alpha(\lambda_1(G) - \lambda_n(G)) - \lambda_1(G)\lambda_n(G)}{(2\alpha + \lambda_1(G) + \lambda_n(G))(\alpha + \lambda_1(G))}.$$

(i) *If G is positive definite, K is positive semidefinite matrix and α is set to be α^*, then*

$$\beta_{opt} = \begin{cases} \alpha^*, & \alpha^* > \sqrt{\lambda_1(G)\lambda_n(G)}, \\ \tilde{\beta}(\alpha^*), & \text{otherwise}, \end{cases}$$

where β_{opt} is the optimal value of β. Moreover, the corresponding upper bounds are given by

$$\sigma(\alpha^*, \beta_{opt}) = \begin{cases} \frac{\alpha^* - \lambda_n(G)}{\alpha^* + \lambda_n(G)}, & \alpha^* > \sqrt{\lambda_1(G)\lambda_n(G)}, \\ \tilde{\sigma}(\alpha^*), & \text{otherwise}. \end{cases}$$

(ii) *If G is positive semidefinite, K is positive definite matrix, and β is set to be β^*, then*

$$\alpha_{opt} = \beta^* + \frac{1}{2}\lambda_n(K),$$

where α_{opt} is optimal of for α. Moreover,

$$\sigma(\alpha_{opt}, \beta^*) = \begin{cases} \frac{\lambda_1(G) - \beta^*}{\lambda_1(G) + \beta^* + \lambda_n(K)/2}, & \beta^* \leq \tilde{\beta}(\alpha_{opt}), \\ \frac{\beta^*}{\beta^* + \lambda_n(K)/2}, & \beta^* \geq \tilde{\beta}(\alpha_{opt}). \end{cases}$$

(iii) *Suppose that G and K are positive definite matrices and α is set to be the prescribed value α^*. Moreover, let*

$$\Delta = \sqrt{\lambda_n(K)^2 + 4(\lambda_1(G) + \lambda_n(G))\lambda_n(K) + 16\lambda_n(G)\lambda_1(G)},$$

then

$$\beta_{opt} = \begin{cases} \alpha^* - \frac{1}{2}\lambda_n(K), & \alpha^* > \frac{1}{4}(\lambda_n(K) + \sqrt{\Delta}), \\ \tilde{\beta}(\alpha^*), & \text{otherwise}, \end{cases}$$

where β_{opt} is the optimal value of β. In addition, we have

$$\sigma(\alpha^*, \beta_{opt}) = \begin{cases} \frac{\alpha^* - \lambda_n(K)/2 - \lambda_n(G)}{\alpha^* + \lambda_n(G)}, & \alpha^* > \frac{1}{4}(\lambda_n(K) + \sqrt{\Delta}), \\ \tilde{\sigma}(\alpha^*), & \text{otherwise}. \end{cases}$$

Proof. First, for a given $\alpha = \alpha^*$, we find the optimal value of β and the corresponding upper bound in Case (i). For both of the convergence intervals given in Case (i) of

Theorem 2.2, there is $\hat{\beta} > 0$ such that

$$\|T_1\| = \begin{cases} \frac{\lambda_1(G)-\beta}{\lambda_1(G)+\alpha}, & 0 < \beta \leq \hat{\beta}, \\ \frac{\beta-\lambda_n(G)}{\lambda_n(G)+\alpha}, & \hat{\beta} \leq \beta. \end{cases} \tag{21}$$

Since, $\hat{\beta}$ minimizes the upper bound $\sigma(\alpha^*, \beta)$, for such a minimum point, we have

$$\frac{\lambda_1(G) - \hat{\beta}}{\lambda_1(G) + \alpha^*} = \frac{\hat{\beta} - \lambda_n(G)}{\lambda_n(G) + \alpha^*}.$$

Hence,

$$\hat{\beta} = \frac{\alpha^*(\lambda_n(G) + \lambda_1(G)) + 2\lambda_n(G)\lambda_1(G)}{2\alpha^* + \lambda_n(G) + \lambda_1(G)} = \tilde{\beta}(\alpha^*). \tag{22}$$

Here we should mention that the computed $\hat{\beta}$ may not belong to the convergence intervals. Suppose that $\hat{\beta} \geq \alpha^* + 2\lambda_n(G)$. From (22) we have $2\alpha^2 + 4\alpha\lambda_n(G) + 2\lambda_n(G)^2 \leq 0$. Therefore we have $(\alpha + \lambda_n(G))^2 \leq 0$ which is a contradiction to the positive definiteness of the matrix G. Hence we have, $\hat{\beta} < \alpha^* + 2\lambda_n(G)$. Now, it is enough to find a condition under which $\hat{\beta}$ satisfies the second interval given in Case (i) of Theorem 2.2. To do so, we should have $\hat{\beta} \geq \alpha^*$. We have $\hat{\beta} < \alpha^*$ if and only if

$$\frac{\alpha^*(\lambda_n(G) + \lambda_1(G)) + 2\lambda_n(G)\lambda_1(G)}{2\alpha^* + \lambda_n(G) + \lambda_1(G)} < \alpha^*,$$

which is itself equivalent to

$$\alpha^* > \sqrt{\lambda_1(G)\lambda_n(G)}.$$

Hence if $\alpha^* > \sqrt{\lambda_1(G)\lambda_n(G)}$, since the proposed convergence assumption, $\hat{\beta}$ can not be considered as the optimal value. In this, case we have

$$\sigma(\alpha^*, \beta) = \frac{\beta - \lambda_n(G)}{\lambda_n(G) + \alpha^*}.$$

Hence, if $\alpha^* > \sqrt{\lambda_1(G)\lambda_n(G)}$, the optimal value of β in $[\alpha^*, \alpha^* + 2\lambda_n(G))$ should be considered as $\beta = \alpha^*$. The given upper bound is obtained by substituting the computed optimal values in $\sigma(\alpha^*, \beta)$.

For Case (ii), thanks to Case (ii) of Theorem 2.2, without loss of generality, we can choose α such that $\beta < \alpha \leq \beta + \frac{1}{2}\lambda_n(K)$, where β^* is an arbitrary chosen value. It can be easily seen that the minimum value of $\sigma(\alpha, \beta^*)$ can be obtained when $\alpha_{opt} = \beta^* + \lambda_n(K)/2$. In other words, denominators in (21) have their maximum values when α takes the proposed value. Moreover, for given β^*, the upper bound of $\sigma(\alpha, \beta^*)$ is determined for $\beta^* \leq \tilde{\beta}(\alpha_{opt})$ and $\beta^* \geq \tilde{\beta}(\alpha_{opt})$.

To prove case (iii), the convergence interval for β can be chosen as,

$$\alpha^* \leq \beta + \frac{1}{2}\lambda_n(K) < \alpha^* + 2\lambda_n(G) + \frac{1}{2}\lambda_n(K),$$

where α^* is an arbitrary chosen value for α. This bound can be equivalently written as

$$\alpha^* - \frac{1}{2}\lambda_n(K) \leq \beta < \alpha^* + 2\lambda_n(G).$$

Now, similar to the proof of Case (i) the proof of (iii) is completed. ∎

Remark 2 Since matrix H is Hermitian positive definite matrix, for the first step in the HSS, GHSS and TGHSS methods, conjugate gradient (CG) method can be efficiently applied. In the second step of proposed methods, some Krylov subspace methods can be implemented. Because of the operations of matrix-vector multiplications in this step, the generalized minimal residual (GMRES) method can be used to solve given system [34].

3. Numerical experiments

In this section, we consider two illustrative examples to show the effectiveness of the proposed method. In the first example, an image restoration problem is considered and in the second one, the convection-diffusion problem is approximately solved by the TGHSS method. All the computations have been implemented with MATLAB 8.1 software on a PC with Core i7, 2.67GHz CPU and 4.00GB RAM.

The image restoration and the solution of the convection-diffusion equation are two common problems which are investigated in the literature. Image restoration problem arises in many fields of applied sciences such as medical and astronomical imaging, engineering, optical systems and many other areas [13, 18, 23]. To simulate image restoration problem, an image is degraded by using point spread function (PSF). This PSF can be led to blurring matrix which blurs the true image. Moreover, the white Gaussian noise is usually added to the blurred image to complete degradation process. Finally, true image is restored (approximated) by using the known PSF, noise and degraded image and solving a linear system (For more details, see e.g. [18, 21, 29]). Convection-diffusion equation has been widely used to describe transport phenomena which arises in engineering, physics and chemistry [14]. Three dimensional of the proposed equation and approximation of its solution have been considered in various studies (e.g. see [12, 39]). The proposed problem is discretized by central finite difference scheme which leads to solve a system of linear equations [6].

For each of the considered image restoration and convection-diffusion problem, a non-Hermitian positive definite system is arisen [6, 29]. Therefore, the proposed TGHSS method can be applied to solve them.

3.1 *Image restoration problem*

Image restoration is one of the problems which leads to solve a sparse linear system. This problem can be modeled as [18, 21, 23]

$$g = Af + \eta, \tag{23}$$

where f, g and η are n^2-dimensional vectors which show the true image, degraded image and noise, respectively, and A is $n^2 \times n^2$ blurring matrix. Because of convolution process and limitation in field of view (FOV), some assumptions are necessary outside the FOV which called boundary conditions (BCs). Zero, periodic, reflexive, antireflective and mean BCs are used to restore images.

Zero BCs is implemented by considering zeros for the outside of FOV which leads to block Toeplitz with Toeplitz blocks (BTTB) for blurring matrix A. Periodically extended data in outside of FOV in periodic BCs, leads to block circulant with circulant blocks (BCCB) matrix A. For the zero and periodic BCs, matrix-vector multiplications are computed by fast Fourier transforms (FFTs) [21]. For the reflexive BCs, block Toeplitz-plus-Hankel with Toeplitz-plus-Hankel blocks (BTHTHB) matrix A is obtained by reflecting the FOV to outside. For the symmetric PSF, the burring matrix is diagonalized with the 2-dimensional fast cosine transform (DCT-III) [21]. Anti-reflection of FOV data to outside leads block Toeplitz-plus-Hankel-plus-rank-2-correction in antireflective BCs. An adaptive antireflection which preserves $\mathcal{C}^1$ continuity and reduces the ringing effects called Mean BCs. This BCs leads to a blurring matrix with block-Toeplitz-plus-Rank4 with Toeplitz-plus-Rank4-blocks (BTR4TR4B) structure. Perrone in [33] for the antireflective BCs and Zhao et al. in [42] for mean BCs, have introduced the Kronecker product approximations to implement the proposed BCs. In this study, since being symmetric property of PSF is not necessary in Kronecker product approximations method, we have used proposed approximation for antireflective and mean BCs.

The Tikhonov regularization is an effective method to solve ill-conditioned systems. Because of ill-conditioning of the system (23), we have applied the proposed method to solve system (23). It can be seen that the Tikhonov regularization method leads to solve the following system

$$(A^TA + \mu^2 I)f = A^T g, \tag{24}$$

where μ is the regularization parameter associated with the noise level. In this study, we consider $0 < \mu < 1$. The discrepancy principle method [32], the L-curve criterion method [20], and generalized cross validation (GCV) method [16, 21] have been presented to give an approximation to regularization parameter. In this study, the GCV method is used to find regularization parameter. This method is based on finding the minimal point of GCV function

$$G(\mu) = \frac{\|A(A^TA + \mu^2 I)^{-1}A^T g - g\|_2}{(\mathrm{trace}(I - A(A^TA + \mu^2 I)^{-1}))^2}. \tag{25}$$

Finding the exact value of μ which minimizes $G(\mu)$ is a critical task. Hence, we use Kronecker product approximation to find an approximation of proposed argument [33].

The following equivalent system have been presented by Lv et al. [29] to restore images

$$\underbrace{\begin{bmatrix} I & A \\ -A^T & \mu^2 I \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} e \\ f \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} g \\ 0 \end{bmatrix}}_{b}, \tag{26}$$

where T is $2n^2 \times 2n^2$ non-Hermitian positive definite matrix and the additive noise is given by auxiliary variable $e = g - Af$. They used a special Hermitian and skew-Hermitian (SHSS) method by setting $\alpha = 1$ in the second equation of (3) to solve system (26) and restore images. In the similar way, we apply the TGHSS method to solve (26) and the results are compared with those of the HSS, SHSS and GHSS methods. To this end, two matrix splittings of T are considered as follows.

Case I) In this case, we use the matrix splitting

$$T = \begin{bmatrix} (1-\mu^2)I & O \\ O & \mu^2 I \end{bmatrix} + \begin{bmatrix} \mu^2 I & O \\ O & O \end{bmatrix} + \begin{bmatrix} O & A \\ -A^T & O \end{bmatrix} = G + K + S, \tag{27}$$

to implement GHSS (GHSS-I) and TGHSS (TGHSS-I) methods, where O is a zero matrix of appropriate size.

Case II) We consider the splitting

$$T = \mu^2 \begin{bmatrix} I & O \\ O & I \end{bmatrix} + \begin{bmatrix} (1-\mu^2)I & O \\ O & O \end{bmatrix} + \begin{bmatrix} O & A \\ -A^T & O \end{bmatrix} = G + K + S, \qquad (28)$$

to use the GHSS (GHSS-II) and TGHSS (TGHSS-II) methods where O is a zero matrix of appropriate size.

Remark 3 From the structure of the matrices G and K, they are symmetric positive definite and symmetric semidefinite matrix, respectively. Hence, the main assumptions of Theorem 2.2 for the convergence are satisfied.

As Benzi mentioned in [11], no general comparison theorem is possible for the spectral radii associated with the GHSS and HSS iterations. Now, similar to the GHSS method, we can not generally give a theorem or result to split H optimally into G and K such that good convergence performance is guaranteed for the TGHSS method. However, as we mentioned in Section 1, dominance of S can be improved by K. Therefore, K should be chosen in such a way that solving the linear system with $S + K + \beta I$ is expected to be less expensive than the classical HSS scheme where inner iterative solvers are used.

We can now briefly summarize the TGHSS method to restore images as the following algorithm:

`Algorithm 1: The TGHSS method to restore images`

1. `Choose values for maximum number of outer iteration` M `and iteration tolerance` τ. `Moreover, let` $f_0 = g$ `and` $e_0 = g - Af_0$ `as initial guesses.`
2. $r_0 := b - Tx_0$;
3. `For` $k = 0, 1, 2, \ldots,$ `until` $\frac{\|r_k\|_2}{\|r_0\|_2} > \tau$ `or` $k < M$ `Do,`

   `Compute` $x_{k+\frac{1}{2}} := (\alpha I + G)^{-1} \left((\alpha I - S - K)x_k + b\right),$

   `Solve` $(\beta I + S + K)x_{k+1} = (\beta I - G)x_{k+\frac{1}{2}} + b,$ `with GMRES method,`

   $r_{k+1} := b - Tx_{k+1},$
4. `EndDo`

Note that a similar algorithm can be written to implement the HSS, SHSS and GHSS methods. Moreover, the structure of the matrix G (or Hermitian part of matrix A for the HSS and SHSS methods) in the proposed image restoration problem shows that the matrix $(\alpha I + G)^{-1}$ can be calculated by straightforward computations.

Example 1 In this example, a simulated MRI of a human brain is used from MATLAB image processing toolbox. We consider the 15th 128×128 horizontal slice from an MRI data scan of a human cranium. Function *psfDefocus* with $dim = 9$ and $R = 4$ is applied as an out-of-focus blur to degrade the proposed image [21]. Moreover, blurred and noisy image is obtained by adding 0.02 Gaussian white noise to the blurred data. The true image and noisy-blurred image are shown in Fig. 1. Note that for the noisy-blurred image we have PSNR = 32.40.

To compare results and to measure the quality of the restored image, the improvement signal-to-noise ratio (ISNR) is used which is defined as follows

$$\text{ISNR} = 10 \cdot \log_{10} \left(\frac{\sum_{i,j} [f(i,j) - g(i,j)]^2}{\sum_{i,j} [f(i,j) - f_{\text{res}}(i,j)]^2} \right),$$

where f, g and f_{res} are the true, degraded and restored images, respectively. The peak signal-to-noise ratio (PSNR) are also considered as follows

$$\text{PSNR} = 10 \log_{10} \frac{4 \times 255^2 \times n^4}{\|f_{\text{res}} - f\|_2^2}$$

where the image is considered as an $n \times n$ matrix. Moreover, the relative error is given by $\|f_{\text{res}} - f\|_2 / \|f\|_2$.

To obtain optimal parameters for the HSS, GHSS and TGHSS methods, we have numerically solved the proposed problem with various values and found best values for the unknowns parameters of proposed methods. For optimal parameter in the SHSS method, we have used

$$\alpha = \frac{\sigma_1^2 + \sigma_n^2 + 2\sigma_1^2 \sigma_n^2}{2 + \sigma_1^2 + \sigma_n^2},$$

where σ_1 and σ_n are the minimum and maximum singular values of A [29]. Note that finding the singular values of A is a critical task. Lv et al. have used the optimal Kronecker product approximation and found the optimal values for α in SHSS method. To implement GMRES method, convergence tolerance is considered to be 10^{-6}. Moreover, dimension of Krylov subspace and maximum number of iterations are set to be $m = 15$ and $itmax = 15$, respectively. Number of outer iteration M and tolerance τ in Algorithm 1 for HSS, SHSS, GHSS and TGHSS methods are also considered to be 15 and 10^{-6}, respectively. The considered values of unknown parameters for the various methods are given in Table 1. To compare the proposed methods, PSNR, ISNR, relative error and the CPU times are given in Tables 2-5. Moreover, restored images with various methods for periodic BCs are shown in Fig. 2. As the numerical results show, the TGHSS method is more accurate than the HSS and SHSS methods. Moreover, based on the used splitting, the applicability of TGHSS method can be different. Considering PSNR, ISNR and the relative error we see that the accuracy of the TGHSS-I method is close to that of the GHSS-I method, but the CPU time of the TGHSS-I method is approximately half of the GHSS-I method. Similarly, although the CPU time of TGHSS-II is more than GHSS-II, the accuracy of the TGHSS-II is more than the GHSS-II. Note that, in image restoration problem, PSNR, ISNR and the relative error are the most important quantities to measure the accuracy and efficiency of the method. Therefore, in comparison with the other methods, specially the SHSS and HSS methods, the TGHSS method is more reliable.

In Tables 6-9, the parameter β (α) is set to be a prescribed value and the relative errors of TGHSS-I and TGHSS-II are given for various values of the parameter α (β). It is mentioned that we fixed α (β) close to its optimal value for five BCs (an average of them). As seen, there is no significant difference between the relative errors of TGHSS-I and TGHSS-II methods for various values of parameters.

Now, we investigate the convergence speed of the proposed methods for periodic and reflexive BCs. The residual error in the kth outer iteration of Algorithm 1 can be defined as $e_k = g - Af_k$. In Fig. 3, $\|e_k\|_2 / \|e_0\|_2$ versus the outer production number k has been plotted. The approximated optimal values for unknown iteration parameters in the proposed methods have been used to plot Fig. 3. As seen, the convergence of the TGHSS-

Table 1. Values of (α, β) for various methods in Example 1.

method \ BC	Zero	Periodic	Reflexive	Antireflective	Mean
HSS	$(2.01, -)$	$(2.01, -)$	$(2.01, -)$	$(2.01, -)$	$(1.07, -)$
SHSS	$(0.3377, -)$	$(0.3333, -)$	$(0.3383, -)$	$(0.6039, -)$	$(0.86, -)$
GHSS-I	$(0.11, -)$	$(0.12, -)$	$(0.09, -)$	$(0.09, -)$	$(0.10, -)$
TGHSS-I	$(0.07, 0.29)$	$(0.08, 0.35)$	$(0.07, 0.22)$	$(0.05, 0.29)$	$(0.07, 0.33)$
GHSS-II	$(0.65, -)$	$(0.65, -)$	$(0.65, -)$	$(0.65, -)$	$(2.21, -)$
TGHSS-II	$(0.33, 0.16)$	$(0.34, 0.14)$	$(0.34, 0.17)$	$(0.36, 0.18)$	$(0.35, 0.16)$

Table 2. PSNR of the various methods for Example 1.

method \ BC	Zero	Periodic	Reflexive	Antireflective	Mean
HSS	33.41	33.44	33.42	33.40	25.40
SHSS	35.03	35.18	35.02	34.58	33.85
GHSS-I	35.78	35.96	35.73	35.64	34.60
TGHSS-I	35.80	35.97	35.74	35.66	34.61
GHSS-II	34.85	34.92	34.85	34.83	33.44
TGHSS-II	35.77	35.94	35.71	35.66	34.53

Table 3. ISNR of the various methods for Example 1.

method \ BC	Zero	Periodic	Reflexive	Antireflective	Mean
HSS	1.02	1.04	1.02	1.00	-7.00
SHSS	2.63	2.78	2.62	2.19	1.45
GHSS-I	3.39	3.56	3.33	3.24	2.20
TGHSS-I	3.40	3.57	3.34	3.27	2.22
GHSS-II	2.45	2.53	2.45	2.43	1.04
TGHSS-II	3.37	3.54	3.31	3.26	2.13

Table 4. Relative error of the various methods for Example 1.

method \ BC	Zero	Periodic	Reflexive	Antireflective	Mean
HSS	0.2808	0.2800	0.2805	0.2812	0.7063
SHSS	0.2329	0.2292	0.2333	0.2453	0.2670
GHSS-I	0.2137	0.2095	0.2151	0.2172	0.2450
TGHSS-I	0.2134	0.2093	0.2149	0.2167	0.2446
GHSS-II	0.2380	0.2360	0.2380	0.2386	0.2800
TGHSS-II	0.2141	0.2099	0.2157	0.2168	0.2470

I method is approximately as fast as GHSS-I while the TGHSS-II method is more faster than the GHSS-II method.

3.2 *Convection-diffusion problem*

Consider the three-dimensional convection-diffusion equation

$$-(u_{xx} + u_{yy} + u_{zz}) + q(u_x + u_y + u_z) + pu = f(x, y, z) \ \text{ in } \ \Omega, \tag{29}$$

Table 5. Time of the various methods for Example 1.

method \ BC	Zero	Periodic	Reflexive	Antireflective	Mean
HSS	9.69	9.48	9.49	9.52	10.52
SHSS	9.70	9.51	9.43	9.65	10.72
GHSS-I	50.02	51.07	60.36	61.61	66.92
TGHSS-I	25.77	25.62	33.67	26.92	25.84
GHSS-II	10.70	9.41	9.48	9.74	5.49
TGHSS-II	17.59	17.51	17.89	18.39	20.60

Table 6. Relative error of the TGHSS-I for $\beta = 0.3$ and various values of α in Example 1.

$\alpha \setminus$ BC	Zero	Periodic	Reflexive	Antireflective	Mean
0.05	0.2148	0.2135	0.2161	0.2168	0.2456
0.07	0.2140	0.2106	0.2156	0.2169	0.2446
0.09	0.2146	0.2104	0.2159	0.2175	0.2449
0.11	0.2153	0.2109	0.2166	0.2182	0.2454
0.13	0.2161	0.2116	0.2174	0.2190	0.2460
0.15	0.2171	0.2125	0.2182	0.2199	0.2467

Table 7. Relative error of the TGHSS-I for $\alpha = 0.07$ and various values of β in Example 1.

$\beta \setminus$ BC	Zero	Periodic	Reflexive	Antireflective	Mean
0.25	0.2142	0.2106	0.2156	0.2169	0.2446
0.27	0.2140	0.2106	0.2156	0.2169	0.2446
0.29	0.2134	0.2106	0.2156	0.2169	0.2446
0.31	0.2141	0.2105	0.2156	0.2170	0.2446
0.33	0.2142	0.2105	0.2157	0.2170	0.2446
0.35	0.2145	0.2113	0.2159	0.2171	0.2465

Table 8. Relative error of the TGHSS-II for $\beta = 0.16$ and various values of α in Example 1.

$\alpha \setminus$ BC	Zero	Periodic	Reflexive	Antireflective	Mean
0.30	0.2155	0.2115	0.2171	0.2177	0.2531
0.32	0.2150	0.2110	0.2165	0.2170	0.2474
0.34	0.2150	0.2110	0.2164	0.2170	0.2470
0.36	0.2150	0.2111	0.2165	0.2170	0.2470
0.38	0.2150	0.2112	0.2165	0.2170	0.2470
0.40	0.2151	0.2113	0.2166	0.2170	0.2471

Table 9. Relative error of the TGHSS-II for $\alpha = 0.34$ and various values of β in Example 1.

$\beta \setminus$ BC	Zero	Periodic	Reflexive	Antireflective	Mean
0.10	0.2208	0.2137	0.2232	0.2276	0.2515
0.12	0.2169	0.2115	0.2189	0.2211	0.2485
0.14	0.2153	0.2099	0.2170	0.2182	0.2473
0.16	0.2150	0.2110	0.2164	0.2170	0.2470
0.18	0.2153	0.2117	0.2166	0.2169	0.2503
0.20	0.2189	0.2156	0.2201	0.2203	0.2674

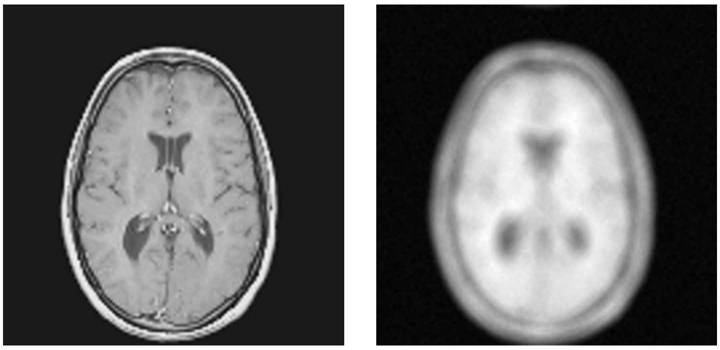
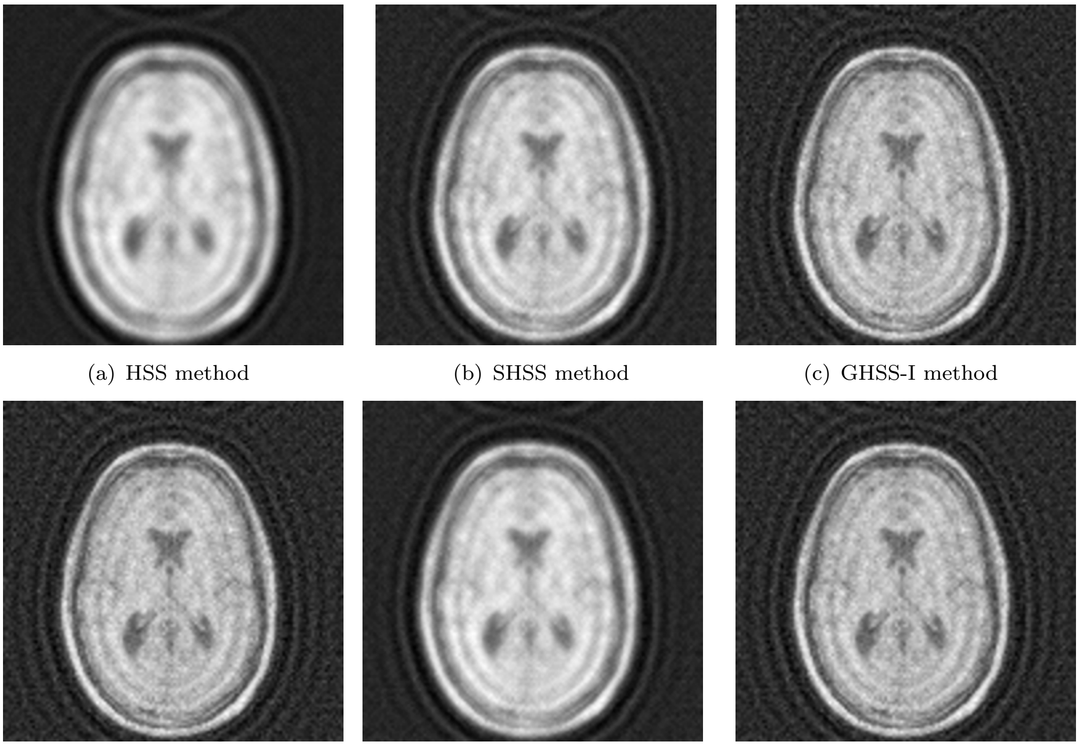


Figure 1. True image (left) and noisy-blurred image (right) for Example 1.



(a) HSS method (b) SHSS method (c) GHSS-I method

(d) TGHSS-I method (e) GHSS-II method (f) TGHSS-II method

Figure 2. Restored image with various methods for periodic BCs in Example 1

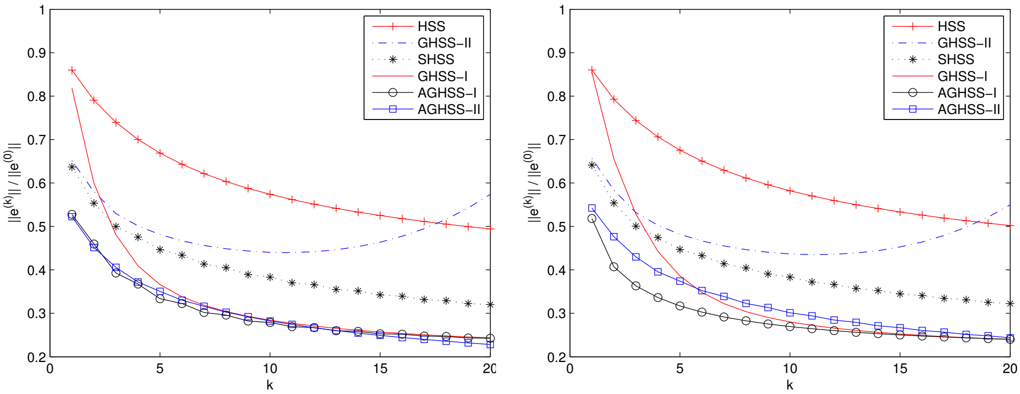


Figure 3. Comparison between residual errors for periodic (left) and reflexive (right) BCs in Example 1.

where q and p are constant, and Ω is the unit cube $[0,1] \times [0,1] \times [0,1]$ with the homogeneous Dirichlet boundary conditions. Discretization of this equation on a uniform grid with $n \times n \times n$ interior nodes (with step size $h = \frac{1}{n+1}$), by using the second order centered differences for the second and first order differentials, gives a linear system of equations of order n^3. This equation can be written as [6]

$$Ax = b, \tag{30}$$

where

$$A = T_x \otimes I \otimes I + I \otimes T_y \otimes I + I \otimes I \otimes T_z + I \otimes I \otimes I, \tag{31}$$

in which

$$T_x = \texttt{tridiag}(c_2, c_1, c_3), \quad T_y = \texttt{tridiag}(c_2, 0, c_3), \quad T_z = \texttt{tridiag}(c_2, 0, c_3),$$

with $c_1 = 6$, $c_2 = -1 - r$, $c_3 = 1 + r$ and $r = qh/2$ (mesh Reynolds number, see, e.g. [19]). For the coefficient matrix in (30), the Hermitian and skew-Hermitian parts are respectively given by

$$H = H_x \otimes I \otimes I + I \otimes H_y \otimes I + I \otimes I \otimes H_z + I \otimes I \otimes I, \tag{32}$$

and

$$S = S_x \otimes I \otimes I + I \otimes S_y \otimes I + I \otimes I \otimes S_z, \tag{33}$$

where

$$H_x = \texttt{tridiag}(\frac{c_2 + c_3}{2}, c_1, \frac{c_2 + c_3}{2}), \;\; H_y = H_z = \texttt{tridiag}(\frac{c_2 + c_3}{2}, 0, \frac{c_2 + c_3}{2}),$$
$$S_p = \texttt{tridiag}(\frac{c_2 - c_3}{2}, 0, -\frac{c_2 - c_3}{2}), \quad p \in \{x, y, z\}.$$

Moreover, centered difference schemes discretization for the diffusion part ($p = q = 0$), gives the following matrix [31]

$$L = A = T \otimes I \otimes I + I \otimes T \otimes I + I \otimes I \otimes T + I \otimes I \otimes I, \tag{34}$$

where

$$T = \texttt{tridiag}(-1, 2, -1). \tag{35}$$

In the next example, we use the proposed matrices H, S and L to implement the HSS, GHSS and TGHSS methods as a preconditioner.

Example 2 In this example, we consider the convection-diffusion problem (29). To implement GHSS and TGHSS methods, the following splitting is presented

$$A = H + S = L + (H - L) + S = G + K + S, \tag{36}$$

where $G = L$ is symmetric strictly dominant matrix with positive diagonal entries, and as a result G is a symmetric positive definite matrix. In addition, $K = H - L$ is symmetric diagonally dominant matrix with positive diagonal entries, and therefore, K is positive semidefinite matrix.

To investigate the efficiency of the proposed methods, the HSS, GHSS and TGHSS methods are used as preconditioners. Note that several papers have been presented to investigate the optimal values of α in the HSS and preconditioned HSS (PHSS) methods. Bai, Golub and Ng [6] presented the optimal value of α for the HSS method that minimizes an upper bound of spectral radius of iteration matrix. The optimal value of α in the HSS has been estimated when A is two-by-two block matrices in [5]. Moreover, Bai [2] implemented the AHSS method for saddle-point problems and found a good approximation of parameters to minimize the upper bound of spectral radius. Recently, a new strategy has been presented to estimate the optimal values of the HSS method [22]. This strategy uses some trace of matrices and estimates an optimal value for α. It seems

Table 10. Approximated optimal values of (α, β) in the TGHSS method for $n = 4$ in Example 2.

(p, q)	(α, β)	$\lambda_n(G)$	Total inner
$(0.01, 1)$	$(0.02, 1.69)$	1.1459	210
$(0.01, 100)$	$(7.01, 7.15)$	1.1459	300
$(0.01, 1000)$	$(17.18, 17.57)$	1.1459	420

that this method requires less computation than other known methods. Bai, Golub and Pan [9] have also investigated the optimal values of the PHSS method. However, finding a good approximation of the optimal values for any problem is so complicated and cost of operations is not affordable. Therefore, similar to image restoration problem, the unknown parameters for iteration methods are experimentally determined. Finding the exact solution of linear systems with coefficient $\alpha I + H$ and $\alpha I + S$ are costly and impractical in implementations. Hence, Bai, Golub and Ng [6, 8] presented the IHSS method to solve the proposed linear systems. Since the $\alpha I + H$ is Hermitian positive definite, the CG method can be effectively used to solve the linear system with the proposed coefficient matrix. Moreover, for the linear system with coefficient matrix $\alpha I + S$, some Krylov subspace methods such as GMRES and Lanczos can be used. Similar to the Benzi's paper [11], we use full GMRES method as outer iteration solver. Tolerance of the outer iteration is considered 10^{-6}. Moreover, preconditioned CG (PCG) method is used to approximate the solution of the obtained symmetric positive definite linear system in each outer iteration. To solve the non-symmetric linear system in each step of outer iteration, the GMRES method is used. The convergence tolerance of the both inner iteration solvers was also set to 10^{-6}. To implement PCG and GMRES, the incomplete Cholesky factorization and incomplete LU (ILU) factorization are used as a preconditioners. The drop tolerance for both of the preconditioners have been set to be 0.01. In Table 10, the approximated optimal values of unknown parameters in the TGHSS method for $p = 0.01$ and $q = 1, 100, 1000$ have been presented. Moreover an approximation of the smallest singular value of G and total number of inner iterations are also given in Table 10. It can be easily seen that the convergence condition is satisfied i.e $\alpha < \beta \leq \alpha + 2\lambda_n(G)$. Iterations number for outer preconditioned GMRES, average number of iterations for inner PCG and ILU-GMRES and total number of inner iterations are given in Tables 11-16 for $p = 0.01, 0.1$ and $q = 1, 10, 1000$. As seen, the outer iteration and total inner iteration is reduced in the TGHSS method. Therefore, CPU time of calculation is less than two other methods. In fact comparison between HSS, GHSS and TGHSS methods shows that the TGHSS method is more effective and applicable method.

In Table 17, the results of the TGHSS method are given for $\alpha = 0.01$, $p = 0.01$, $q = 1$ and $n = 64$ for several values of β. A similar investigation is given in Table 18 by fixing β. As this results show, small changes in the parameters lead to slight changes in the final results.

For more investigation, we set $n = 128$, $p = 0.01$ and $q = 1, 10$. Moreover, the tolerance of the inner iteration solvers is decreased to 10^{-2} to save the CPU time. As the numerical results show in the Tables 19-20, the number of outer iterations and the CPU time of TGHSS method has been decreased in comparison with HSS and GHSS methods which is indicative of the effectiveness of the proposed method.

Remark 4 Note that for the larger values of p, the linear system $Ax = b$ will be more well-conditioned. Therefore, as we tested numerically, approximately the same results are given by the HSS, GHSS and TGHSS methods for larger values of p.

Table 11. Comparison between various methods for $p = 0.01$, $q = 1$ and $n = 64$ in Example 2.

Preconditioner	(α, β)	Outer its.	PCG /outer	GMRES/outer	Total inner	CPU time
HSS	$(0.10, -)$	6	87	18	630	40.33
GHSS	$(0.04, -)$	5	84	21	525	33.44
TGHSS	$(0.01, 0.39)$	3	66	7	219	13.29

Table 12. Comparison between various methods for $p = 0.01$, $q = 100$ and $n = 64$ in Example 2.

Preconditioner	(α, β)	Outer its.	PCG /outer	GMRES/outer	Total inner	CPU time
HSS	$(1.69, -)$	11	43	71	1254	132.65
GHSS	$(1.68, -)$	11	43	71	1254	131.50
TGHSS	$(1.60, 1.68)$	10	41	66	1070	111.35

Table 13. Comparison between various methods for $p = 0.01$, $q = 1000$ and $n = 64$ in Example 2.

Preconditioner	(α, β)	Outer its.	PCG /outer	GMRES/outer	Total inner	CPU time
HSS	$(14.62, -)$	13	24	81	1365	166.17
GHSS	$(14.51, -)$	13	24	81	1365	166.47
TGHSS	$(14.53, 14.60)$	12	23	76	1188	143.49

Table 14. Comparison between various methods for $p = 0.1$, $q = 1$ and $n = 64$ in Example 2.

Preconditioner	(α, β)	Outer its.	PCG /outer	GMRES/outer	Total inner	CPU time
HSS	$(0.13, -)$	4	55	11	264	18.19
GHSS	$(0.10, -)$	3	61	10	213	13.26
TGHSS	$(0.10, 0.36)$	2	58	6	128	7.84

4. Conclusion

In this paper, a two-parameter version of the GHSS method (TGHSS) has been presented by adding a new parameter to the GHSS method. Then, we have investigated the convergence properties of the method. Image restoration and convection-diffusion problems

Table 15. Comparison between various methods for $p = 0.1$, $q = 100$ and $n = 64$ in Example 2.

Preconditioner	(α, β)	Outer its.	PCG /outer	GMRES/outer	Total inner	CPU time
HSS	$(2.05, -)$	9	33	61	846	90.84
GHSS	$(2.10, -)$	9	34	61	855	91.17
TGHSS	$(1.46, 1.59)$	8	41	56	776	76.13

Table 16. Comparison between various methods for $p = 0.1$, $q = 1000$ and $n = 64$ in Example 2.

Preconditioner	(α, β)	Outer its.	PCG /outer	GMRES/outer	Total inner	CPU time
HSS	$(16.02, -)$	13	24	81	1365	165.31
GHSS	$(14.70, -)$	13	24	81	1365	165.91
TGHSS	$(14.64, 14.70)$	12	23	76	1188	143.03

Table 17. Results of the TGHSS method for $\alpha = 0.01$, $p = 0.01$, $q = 1$ and $n = 64$ in Example 2.

β	Outer its.	PCG /outer	GMRES/outer	Total inner	CPU time
0.20	3	66	10	228	13.99
0.25	3	66	9	225	13.86
0.30	3	66	8	222	13.55
0.35	3	66	7	219	13.36
0.40	3	66	7	219	13.31
0.45	3	66	7	219	13.35

Table 18. Results of the TGHSS method for $\beta = 0.39$, $p = 0.01$, $q = 1$ and $n = 64$ in Example 2.

α	Outer its.	PCG /outer	GMRES/outer	Total inner	CPU time
0.05	5	84	8	460	28.28
0.10	6	92	9	606	37.62
0.15	7	89	10	693	45.16
0.20	7	78	10	616	42.35
0.25	8	78	11	712	50.11
0.30	8	71	11	656	48.64

Table 19. Comparison between various methods for $p = 0.01$, $q = 1$ and $n = 128$ in Example 2.

Preconditioner	(α, β)	Outer its.	PCG /outer	GMRES/outer	Total inner	CPU time
HSS	$(0.05, -)$	7	34	9	301	228.64
GHSS	$(0.01, -)$	6	47	9	336	212.51
TGHSS	$(0.01, 0.1)$	3	58	3	183	91.93

Table 20. Comparison between various methods for $p = 0.01$, $q = 10$ and $n = 128$ in Example 2.

Preconditioner	(α, β)	Outer its.	PCG /outer	GMRES/outer	Total inner	CPU time
HSS	$(0.13, -)$	12	25	29	648	644.87
GHSS	$(0.16, -)$	12	26	22	576	579.14
TGHSS	$(0.14, 0.38)$	11	29	14	473	483.40

have been also considered to investigate the effectiveness and accuracy of the method. As the numerical results show, the TGHSS method is more effective than the GHSS and HSS methods for the examples we have considered.

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